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PLANE AND SPHERICAL

TRIGONOMETRY

BY

ALFRED H. WELSH, A.M.
(*Ohio State University*)

AUTHOR OF DEVELOPMENT OF ENGLISH LITERATURE AND LANGUAGE,
ESSENTIALS OF ENGLISH, COMPLETE RHETORIC, ESSENTIALS OF
GEOMETRY, MAN AND HIS RELATIONS, FIRST LESSONS
IN ENGLISH, LESSONS IN ENGLISH GRAMMAR

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TO

My Friend and Former Colleague,

DR. E. FRAUNFELTER,

WISDOM-LOVING AND CAPACIOUS AS A MAN,
ALERT AND INVENTIVE AS AN EDUCATOR,

THIS WORK IS INSCRIBED

WITH THE AUTHOR'S SINCERE REGARDS AND PLEASANT
RECOLLECTIONS OF FELLOWSHIP.

PREFACE.

THE aim of this work is to furnish just so much of PLANE AND SPHERICAL TRIGONOMETRY as will suffice for the demands of our best schools and colleges. It is designed to be neither too elementary nor too voluminous, and yet to embody the latest developments of the science and the most approved modern methods of instruction. The work is believed to be adapted to the ability of the average pupil, and it is hoped will prove both intelligible and attractive to all, — thus providing a discipline for the mind, and conducing to a delightful expertness hardly to be obtained in any other manner.

CHAPTER I treats of the nature and use of Logarithms. It contains necessary theorems, problems, corollaries, and scholiums, together with a sufficient variety of exercises to insure a speedy and thorough mastery of the subject.

CHAPTER II treats of the solution of Plane Triangles. In discussing the Trigonometric Functions, the author has adopted the method now in such general favor in Europe, — the method of *ratios*. The actual experience of the class-room has proved that students trained in this method get hold of the applications of trigonometry to the physical and mechanical sciences much more easily and readily than others trained in the old, which, in addition to being so much more unwieldy, involves so much greater expenditure of time and labor. An explanation of the line-method is added, however, for the convenience of those who insist upon its use.

The trigonometric functions are defined, at the outset, only with reference to the acute angle; nor is the student allowed to suspect that such definitions are not final and general until there is an absolute necessity for their extension. The true order of development, the natural order in which the individual mind will receive science, is from particulars to generals. *Little by little* is the golden rule of Nature, and should be the canon of the teacher's art.

The subject of Natural Functions has received unusual care, and the solution of triangles by means of them will form, it is thought, a valuable feature of the work.

CHAPTER III investigates the properties and the relations of the Trigonometric Functions as applied to angular magnitude in general. It is hoped that the subject has been relieved of its usual irksomeness by such lucid explanations and simple methods as will make it easily apprehensible and pleasurable, while it induces and requires a careful and continuous attention. Practical exercises, including the construction of angles and the solution of equations, are given at frequent intervals, to fix the principles in the mind of the student, as well as to illustrate their bearing and utility. Pains have been taken so to graduate the exercises and to accompany them by solutions, hints, and suggestions, here and elsewhere, that they shall discourage none, while they stimulate the interest and earnestness of all.

CHAPTER IV contains those principles of Spherical Trigonometry which are most important on account either of their applications, or of their connection with other parts of a mathematical course. It is believed that the treatment will be found to conform in every particular to the latest and best methods of exposition.

The limits of a preface will not permit a statement of all the peculiarities of the book; nor is such a statement necessary, since those who are interested to know will examine for themselves. It will be noticed, however, as a pervading feature, that important principles, when possible, have been reduced to the form of theorems, or rules, which are expressed in italics and designed for memorization; also, that the student is required to translate formulæ into ordinary language, — an exercise adopted the method now in such general favor in Europe, — the method of *ratios*. The actual experience of the class-room has proved that students trained in this method get hold of the applications of trigonometry to the physical and mechanical sciences much more easily and readily than others trained in the old, which, in addition to being so much more unwieldy, involves so much greater expenditure of time and labor. An explanation of the line-method is added, however, for the convenience of those who insist upon its use.

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of great service in educating the faculty of abstraction and generalization.

The sources from which material has been drawn are various. In particular, the works of DeMorgan, Jardine, Todhunter, Wheeler, Loomis, Chauvenet, Olney, Schuyler, and Wentworth have contributed much in matter and form.

A. H. W.

Columbus, Ohio, Jan. 10, 1888.

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PLANE AND SPHERICAL TRIGONOMETRY.

CHAPTER I.

INTRODUCTION. — LOGARITHMS.

§ 1. DEFINITIONS.

1. The **Logarithm** of a number is the *exponent* denoting the power to which a fixed number must be raised in order to produce the given number.

Thus, if 4 be the given number and 2 the fixed number, the *logarithm* of 4 is 2; since 2 must be raised to the second power to produce 4. If 8 be the given number, referred to the same fixed number, its *logarithm* is 3; since $2^3 = 8$. If 10 be the fixed number and 1000 the given number, the *logarithm* of 1000 is 3; since $10^3 = 1000$. If 16 be the fixed number and 4 the given number, the *logarithm* of 4 is $\frac{1}{2}$, or .5; since $16^{\frac{1}{2}} = 16^{.5} = \sqrt{16} = 4$. If 8 be the fixed number and 4 the given number, the *logarithm* of 4 is $\frac{2}{3}$, or .666+; since

$$8^{\frac{2}{3}} = 8^{.666+} = \sqrt[3]{8^2} = 4.$$

2. The *fixed* number is called the **Base**. The base in most frequent use is 10, and the logarithms of numbers referred to this base are called *Common Logarithms*. Hence,

3. The **Common Logarithm** of a number is the *exponent* denoting the power to which 10 must be raised to produce the number.

Thus, if 5.8279 = the common logarithm of 672887, the meaning is, that $10^{5.8279} = 672887$.

In speaking of the logarithms of numbers, the common logarithm is always meant, unless the contrary is stated. *Log* is prefixed to numbers as a convenient abbreviation for *logarithm*.

§ 2. GENERAL PRINCIPLES.

THEOREM I.

If a number is an exact power of ten, its logarithm is integral.

For,

$$\begin{array}{rrll}
10^0 = & 1; & \therefore \text{ by definition,} & \log 1 = 0. \\
10^1 = & 10; & \therefore \quad \text{"} \quad \text{"} & \log 10 = 1. \\
10^2 = & 100; & \therefore \quad \text{"} \quad \text{"} & \log 100 = 2. \\
10^3 = & 1000; & \therefore \quad \text{"} \quad \text{"} & \log 1000 = 3. \\
\cdot & \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \\
10^p = & n; & \therefore \text{ by definition,} & \log n = p,
\end{array}$$

n denoting *any* exact power of 10.

Cor. I. — *The logarithm of a number between* 1 *and* 10 *is a decimal.*

For, $\log 1 = 0$, and $\log 10 = 1$; hence, if the number is greater than 1 and less than 10, its logarithm is greater than 0 and less than 1, and therefore is a fraction or *deci-*

mal. Thus, it is found that the logarithm of 7 is .8451; that is, $10^{.8451} = 7$.

Cor. II. — *The logarithm of a number between* 10 *and* 100 *is* 1 + *a decimal.*

For, log 10 = 1, and log 100 = 2; hence, if the number is greater than 10 and less than 100, its logarithm is greater than 1 and less than 2, and therefore is 1 *plus* a fraction, or 1 + a *decimal.* Thus, it is found that the logarithm of 88 is 1.9445; that is, $10^{1.9445} = 88$.

Cor. III. — *The logarithm of a number between* 100 *and* 1000 *is* 2 + *a decimal.*

For, log 100 = 2, and log 1000 = 3; hence, if the number is between 100 and 1000, the logarithm is between 2 and 3, and therefore is 2 + a decimal.

Thus, it is found that the logarithm of 234 is 2.3692; that is, $10^{2.3692} = 234$.

Cor. IV. — *In general, if a number is not an exact power of* 10, *its logarithm will consist of two parts, — an entire part and a decimal part.*

The entire part is called the **Characteristic**; the decimal part, the **Mantissa.**

§ 3. LAWS OF THE CHARACTERISTIC.

THEOREM II.

The logarithm of the quotient of any number divided by 10, *is* 1 *less than the logarithm of the number.*

Let (1) $10^x = n;$

then, by definition, $\log n = x.$

Also (2) $10^1 = 10;$ whence, by definition,

$$\log 10 = 1.$$

$$(1) \div (2) = (3) \quad 10^{x-1} = \frac{n}{10}; \text{ whence, by definition,}$$

$$\log \frac{n}{10} = x - 1; \text{ that is,}$$

the logarithm of *any number divided by 10* is one less than the logarithm of the number.

THEOREM III.

If the number is integral or mixed, the characteristic is positive, and is 1 *less than the number of integral places.*

Suppose it to be known that

$\log 672887 = 5.8279.$ Dividing by 10,

successively, $\log 67288.7 = 4.8279$ . . . . . *Th. II.*

$\log 6728.87 = 3.8279$ . . . . . "

$\log 672.887 = 2.8279$ . . . . . "

$\log 67.2887 = 1.8279$ . . . . . "

$\log 6.72887 = 0.8279$ . . . . . "

It is clear that the characteristic of the logarithm of 672887 should be, as it is, 5; for this number is between 100000 and 1000000:

but $\log 100000 = 5$, since $10^5 = 100000$,

and $\log 1000000 = 6$, since $10^6 = 1000000$;

hence, the logarithm of 672887 is *between* 5 and 6, and therefore is 5 + a decimal.

THEOREM IV.

If the number is entirely decimal, the characteristic is negative and is, numerically, 1 *greater than the number of ciphers immediately following the decimal point.*

From Theorem III, log 6.72887 = 0.8279. Dividing 6.72887 successively by 10 and diminishing the logarithm by units for each division according to Theorem II, we have

$$\begin{aligned}
\log\ 0.672887 &= -0.1721 = \bar{1}.8279\\
\log\ 0.0672887 &= -1.1721 = \bar{2}.8279\\
\log\ 0.00672887 &= -1.1721 = \bar{3}.8279\\
\log\ 0.000672887 &= -1.1721 = \bar{4}.8279
\end{aligned}$$

It will thus be observed that the logarithms of decimals may be written in two different ways. To avoid the necessity of a special table for the logarithms of decimals, and for other good reasons, mathematicians have agreed to write the logarithms of decimals as in the last column above. The learner must especially notice that *in this column the minus sign affects only the characteristic*, the mantissa being invariably positive.

It will also be observed that the number of units in any negative characteristic is always greater by one than the number of ciphers immediately following the decimal point.

§ 4. LAW OF THE MANTISSA.

THEOREM V.

The mantissa is the same for all positions of the decimal point.

This is apparent from an inspection of the logarithms under Theorems III and IV. The characteristics differ, but the decimal parts of the logarithms remain unchanged. Thus, the *mantissa* of log .000672887 is the same as the mantissa of log 67.2887, which is the same as the mantissa of log 672887.

Cor. — *The mantissa of a decimal, or of a mixed number, is the same as the mantissa of the number considered as integral.*

§ 5. EXERCISES.

1. What is the characteristic of the logarithm of 5?
2. What is the characteristic of the logarithm of 96?
3. What is the characteristic of the logarithm of 3457896472?
4. What is the characteristic of the logarithm of .6? (See Th. IV.)
5. What is the characteristic of the logarithm of .00476?
6. What is the characteristic of the logarithm of .60458?
7. What is the characteristic of the logarithm of 374.00689?
8. What is the characteristic of the logarithm of $\frac{1}{7}$? (Reduce to a decimal.)
9. If the characteristic of the logarithm of a number is 3, the number has how many integral places?
10. If the characteristic of the logarithm of a number is 0, the number has how many integral places?
11. If the characteristic of the logarithm of a number is $\overline{2}$, is the number integral or decimal?

12. If the characteristic of the logarithm of a number is $\bar{1}$, how many ciphers immediately follow the decimal point? If the characteristic is $\bar{4}$?

13. Three numbers are 7856943, 7.856943, and .00007856943; what is the difference of their mantissas?

§ 6. TABLE OF LOGARITHMS.

A Table of Logarithms is a collection of numbers and their logarithms, so arranged that from it may readily be obtained the logarithm of any number, or the number corresponding to any logarithm. In the following Problems will be explained the

§ 7. MANNER OF USING THE TABLE.

PROBLEM I.

To find the logarithm of any integral number less than 200.

Look for the given number in the column headed 'N.' Opposite this, in the column headed '0,' is the mantissa of the required logarithm. Find the characteristic by Theorem III.

PROBLEM II.

To find the logarithm of any integral number from 200 *to* 2000.

Look for the first two or three figures of the given number in the column headed 'N.' Then pass along the horizontal line until you reach the column headed with the remaining figure of the number, at which place is the mantissa of the logarithm required. Find the characteristic by Theorem III.

Thus, let it be required to find the logarithm of 496. Looking in the 'N' column for 49, and tracing the horizontal line to the '6' column, we find the mantissa 6955, to which prefixing the characteristic, we have

$$\log 496 = 2.6955.$$

PROBLEM III.

To find the logarithm of any integral number greater than 2000.

Find the characteristic by Theorem III.

To find the mantissa, place a decimal point after the fourth figure (Th. V). The given number is thus converted into a mixed number. Find the mantissa of the entire part by Prob. II. Then, from the column headed 'D,' take the corresponding *tabular difference,* multiply it by the decimal part, add the product to the mantissa already found, and the result will be the mantissa required.

Scholium.—If the decimal part of the product to be added is less than .5, reject it; otherwise, add 1 to the entire part.

Thus, let it be required to find the logarithm of 1234567.

Characteristic = 6.
Mantissa of log 1234 = .0913.
Tabular difference = 3;
$3 \times .567 = 2.$

$\therefore$ log 1234567 = 6.0915.

PROBLEM IV.

To find the logarithm of a mixed number.

Find the characteristic by Theorem III. To find the mantissa, consider the number as *integral* (Th. V).

PROBLEM V.

To find the logarithm of a decimal.

Find the characteristic by Theorem IV.

To find the mantissa, drop the decimal point, and, by the preceding problems, find the mantissa of the number considered as *integral.*

Scholium.—Thus, let it be required to find the logarithm of .06101084.

Characteristic = $\bar{2}$ . . . . *Th. IV.*
Mantissa of log 610 = .7853.
Tabular difference = 7; $7 \times .084 =$ 1.

$\therefore$ log .06101084 = $\bar{2}.7854$.

§ 8. EXERCISES.

1. Find the logarithm of 52376.
2. Find the logarithm of 456875.
3. Find the logarithm of 456.875.
4. Find the logarithm of 1.5.
5. Find the logarithm of .00456842.
6. Find the logarithm of $\frac{3}{8}$.
7. Find the logarithm of $\frac{4}{55}$.

PROBLEM VI.

To find the number corresponding to a given logarithm.

Reverse the process by which the logarithm was found. Neglect the characteristic, and look in the '0' column for the mantissa. If the mantissa is not found exactly, continue the search in one of the nine following columns. If the exact mantissa is found, the first two figures of the corresponding number will be found directly opposite in the 'N' column, and the third figure at the top of the page.

If the exact mantissa cannot be found, find the *next less* mantissa, and set aside the corresponding number; subtract this mantissa from the given one; divide the remainder (annexing ciphers, if necessary) by the corresponding *tabular difference*, and annex the quotient to the number already found.

Then, if the characteristic is *positive*, point off (beginning at the left hand) a number of integral places equal to the characteristic plus 1; the result will be the required number.

If the characteristic is *negative*, the required number is a pure decimal, and the number of ciphers to be prefixed (that is, immediately following the decimal point) is one *less* than the number of units in the characteristic.

Scholium.—Thus, let it be required to find the number whose logarithm is 2.7868.

Looking in the '0' column for 78, we find in the column headed '2' the figures 7868; and directly opposite, the number 61. As the characteristic is +2, 612 is the number required.

Let it be required to find the number whose logarithm is 2.5775.

Looking in the '0' column for 57, we find *in the line above*, and in the column headed '8,' the mantissa 5775, and, since the characteristic is +2, the required number is 378.

Let it be required to find the number whose logarithm is $\bar{3}$.0654.

Given mantissa = .0655.
Next less mantissa = .0652.
Corresponding number = 1162.
Difference of mantissas = 3.
Tabular difference = 4.
3 ÷ 4 = .75.

Annexing this quotient 75 to 1162,
and omitting the decimal point,
the result = 116275.

Prefix two ciphers to this result, since the characteristic is $\bar{3}$, and the required number = 0.00116275.

§ 9. EXERCISES.

1. Find the number whose logarithm is 3.953228.
2. Find the number whose logarithm is 1.826075.
3. Find the number whose logarithm is 5.827943.
4. Find the number whose logarithm is 1.467416.
5. Find the number whose logarithm is $\bar{2}.577520$.
6. Find the number whose logarithm is $\bar{4}.496930$.

§ 10. MULTIPLICATION BY LOGARITHMS.

THEOREM VI.

The logarithm of the product of two quantities is equal to the sum of their logarithms.

Let m and n denote any two numbers referred to the base b, and suppose

$$\begin{aligned} &(1) \quad b^x = m, \\ \text{and } &(2) \quad b^y = n; \text{ then, by definition,} \\ &\log m = x, \\ &\text{and } \log n = y. \\ (1) \times (2) = &(3) \quad b^{x+y} = mn; \therefore \text{ by definition,} \\ &\log mn = x + y \\ &= \text{logarithm of } m + \text{logarithm of } n. \end{aligned}$$

Cor. — *The logarithm of the product of any number of quantities is equal to the sum of the logarithms of those quantities.*

Scholium 1.—Hence, to multiply by means of logarithms, we have the

RULE.

Find the logarithms of the factors, and take their sum; then find the number corresponding to the resulting logarithm. This number will be the product required.

Scholium 2.—It is to be observed that the characteristics (which may have unlike signs) are to be added *according to algebraic principles.*

§ 11. EXERCISES.

1. Find the product of 3.902, 597.16, and .0314728.

$$\begin{aligned} \log\ 3.902 &= .5913, \\ \log\ 597.16 &= 2.776, \\ \log\ .0314728 &= \bar{2}.4979, \\ \therefore \log\ \text{product} &= 1.8653, \\ \therefore \text{product} &= 73.3354. \end{aligned}$$

2. Find the product of 85734 by 12.8746.

3. Find the product of 8.5 by .00048.

4. Find the product of .4672 by .00832.

§ 12. DIVISION BY LOGARITHMS.

THEOREM VII.

The logarithm of the quotient of two quantities is equal to the difference of their logarithms.

Let (1) $b^x = m$,

and (2) $b^y = n$; then, by definition,

$$\log m = x,$$

and $\log n = y$.

$$(1) \div (2) = (3)\ b^{x-y} = \frac{m}{n};\ \therefore \text{ by definition,}$$

$$\log \frac{m}{n} = x - y$$

$= \text{logarithm of } m - \text{logarithm of } n.$

Scholium 1.—Hence, to divide by means of logarithms, we have the

RULE.

From the logarithm of the dividend subtract the logarithm of the divisor; find the number corresponding to the resulting logarithm. This number will be the quotient required.

Scholium 2.—Care must be taken to subtract according to algebraic principles.

§ 13. EXERCISES.

1. Divide 357.32514 by .03572514.

$\log 357.32514 = 4.5531.$

$\log .035732514 = \bar{2}.5531.$

$\therefore \log \text{quotient} = 6.000000.$

$\therefore \text{quotient} = 10000.$

2. Divide 1712 by 432.
3. Divide 5536.177 by .0019577.
4. Divide .4762 by .003782.

§ 14. ARITHMETICAL COMPLEMENT.

The **Arithmetical Complement** of the logarithm of a number is the remainder after subtracting the logarithm from 10.

THEOREM VIII.

The difference between two logarithms is equal to the first plus the arithmetical complement of the second, minus 10.

Let $l =$ first logarithm,

$l' =$ second logarithm,

and $a.\,c. =$ arithmetical complement of second;

then $a.\,c. = 10 - l'$, by definition;

$\therefore \quad l' = 10 - a.\,c.$

Hence, $l - l' = l - (10 - a.\,c.) = l + a.\,c. - 10.$

Scholium 1. — Hence, to divide by means of the arithmetical complement, we have the

RULE.

To the logarithm of the dividend add the arithmetical complement of the logarithm of the divisor, and diminish the sum by 10; the number corresponding to the resulting logarithm will be the required quotient.

Scholium 2. — The arithmetical complement of a logarithm is most conveniently obtained by *commencing at the left hand, and subtracting each figure from 9, till the last significant figure is reached, which must be subtracted from 10.*

§ 15. EXERCISES.

1. Given, $48.6 : 762.4 :: 28.64 : x$, to find x.

$$x = \frac{762.4 \times 28.64}{48.6}.$$

$$\begin{aligned} \log x &= \log (762.4 \times 28.64) - \log 48.6. \\ &= \log 762.4 + \log 28.64 + a.\,c.\,\log 48.6 - 10. \end{aligned}$$

. . . *Th. VI, VII, and VIII.*

$$\begin{aligned} \log 762.4 &= 2.8822. \\ \log 28.64 &= 1.4570. \\ a.\,c.\,\log 48.6 &= 8.3133. \end{aligned}$$

Adding and subtracting 10 from the result,

we have $\log x = 2.6525.$

$\therefore \quad x = 449.3.$

2. Divide 483672 by 62.864.
3. Given, $7856432 : 987678 :: 148432 : x$, to find x.
4. Given, $.0486 : x :: .496 : .00872$, to find x.

§ 16. INVOLUTION BY LOGARITHMS.

THEOREM IX.

The logarithm of any power of a quantity is equal to the logarithm of the quantity multiplied by the exponent of the power.

Let (1) $b^x = n;$

then (2) $\log n = x.$

$(1)^p = (3) \quad b^{xp} = n^p.$

$\therefore \quad \log n^p = xp$

= logarithm of the quantity into the exponent of the power.

Scholium.—Hence, to involve a quantity to a given power, by means of logarithms, we have the

RULE.

Find the logarithm of the quantity, and multiply it by the exponent of the power; find the number corresponding to the result. This number will be the power required.

§ 17. EXERCISES.

1. Find the 8th power of 2.14.

$$\begin{aligned} \log 2.14 &= 0.3304 \\ &\underline{8} \\ \therefore \log \text{power} &= 2.6432 \\ \therefore \text{power} &= 439.7+. \end{aligned}$$

2. Find the cube of 28.75.
3. Find the 7th power of .086.
4. Find the 9th power of .5.

§ 18. EVOLUTION BY LOGARITHMS.

THEOREM X.

The logarithm of any root of a quantity is equal to the logarithm of the quantity divided by the index of the root.

Let (1) $b^x = n$;

then $\log n = x$.

$$\sqrt[r]{(1)} = (2)\ b^{\frac{x}{r}} = \sqrt[r]{n};\ \therefore \text{ by definition,}$$

$$\log \sqrt[r]{n} = \frac{x}{r}$$

= logarithm of quantity ÷ by index of root.

Scholium 1. — Hence, to extract a root of a quantity by means of logarithms, we have the

RULE.

Find the logarithm of the quantity, and divide it by the index of the root; find the number corresponding to the result. This number will be the root required.

Scholium 2. — *If the characteristic is negative, and not a multiple of the index, add to it the smallest negative number that will make it a multiple, then prefix the same number (with the plus sign) to the mantissa.*

Thus, log .00000081 = 7.9085.

Now $\bar{7}.9085$ is evidently the same as $\bar{8} + 1.9085$, which, divided by 2, gives $\bar{4}.9542$; divided by 4, gives $\bar{2}.4771$.

It is to be remembered that the characteristic *alone* is negative . . . . *Th. IV, Sch.*

§ 19. EXERCISES.

1. What is the cube root of 86?

$$\log 86 = 1.9345.$$

$$\frac{\log 86}{3} = 0.6448.$$

$$\therefore \text{ cube root } = 4.414+.$$

2. What is the 4th root of 327.6?
3. What is the 5th root of 6?
4. What is the 6th root of .008706?
5. Given, $x = \sqrt[4]{\dfrac{435.6 \times .48}{6782 \times .0005}}$, to find x.

Suggestion. $\text{Log } \dfrac{435.6 \times .48}{6782 \times .0005} = \text{what?}$

6. What is the 6th root of .0006?

CHAPTER II.

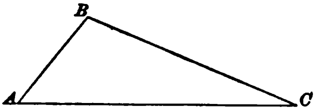
PLANE TRIGONOMETRY.

§ 1. DEFINITIONS.

1. **Plane Trigonometry** is the science which investigates the relations between lines and angles.

2. Its immediate application is the *Solution of Plane Triangles;* that is, the computation of unknown parts of the triangle from known parts.

Thus, given the sides *AC* and *BC*, and their included angle *C*, it is required to compute the length of the side *AB* and the magnitude of the remaining angles, *A* and *B*.



3. A **Line** is represented numerically by the number of times it contains another line assumed as the *linear unit.* Thus, assuming a line one foot long as the linear unit, a line eight feet long is represented numerically by 8.

4. A **Surface** is represented numerically by the number of times it contains the *superficial unit*, — the *square* whose side is the linear unit.

5. A **Solid** is represented numerically by the number of

times it contains the *solid unit,* — the *cube,* each of whose edges is the linear unit.

6. An **Angle** is represented numerically by the number of times it contains the *angular unit,* — the angle of *one degree.*

7. An **Arc** is represented numerically by the number of times it contains the *unit of arc,* — $\frac{1}{90}$th part of a quadrant, also called a *degree.*

8. Since the angle and its intercepted arc are so related that, if one is increased or diminished, the other is increased or diminished in the same ratio, the angle is said to be measured by its intercepted arc. Hence, the angle and the arc are both expressed numerically by the unit degree and its subdivisions, minutes and seconds.

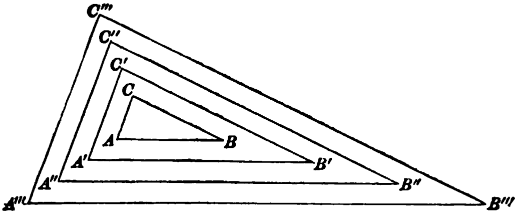
9. Numbers may be generalized; and lines, surfaces, solids, angles, and arcs may be represented by *letters.*

10. The expression of lines and angles by *numbers* — whereby we are enabled to compare one straight line with another, and one angle with another — is the *first* step towards subjecting a triangle to computation. But lines and *angles* are dissimilar quantities.

Hence we must next devise some plan of measuring angles that shall express them by means of *lines,* or of the *ratios* of lines. The comparison of lines with angles will then resolve itself into the comparison of lines with lines, and we shall be able to make calculations in which lines and angles are the *data.*

To this end, let us briefly consider the

§ 2. RELATIONS BETWEEN THE SIDES AND ANGLES OF TRIANGLES.



1. Let ABC be a triangle whose three angles, A, B, and C, are *given*, or *known*. Let $A'B'C'$, $A''B''C''$, etc., be other triangles whose angles are equal to the ***same three given angles.***

Then, since all these triangles are similar,

$$AB : AC :: A'B' : A'C' :: A''B'' : A''C'' :: A'''B''' : A'''C''';$$

$$AB : BC :: A'B' : B'C' :: A''B'' : B''C'' :: A'''B''' : B'''C''';$$

$$BC : AC :: B'C' : A'C' :: B''C'' : A''C'' :: B'''C''' : A'''C''';$$

$$\therefore \quad \frac{AB}{AC} = \frac{A'B'}{A'C'} = \frac{A''B''}{A''C''} = \frac{A'''B'''}{A'''C'''};$$

$$\frac{AB}{BC} = \frac{A'B'}{B'C'} = \frac{A''B''}{B''C''} = \frac{A'''B'''}{B'''C'''};$$

$$\frac{BC}{AC} = \frac{B'C'}{A'C'} = \frac{B''C''}{A''C''} = \frac{B'''C'''}{A'''C'''}.$$ Hence,

(1.) So long as the angles are unchanged, though the absolute magnitude of the sides may vary, the *ratios* of the sides are the same.

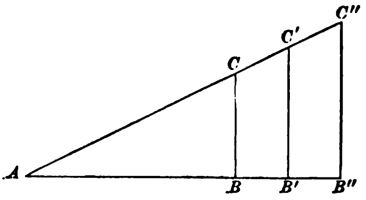
(2.) If the angles change, the *ratios* change; for, if they do not change, the triangles will be similar and the angles will be equal, which is contrary to the hypothesis.

(3.) Conversely, so long as the *ratios* of the sides are unchanged, whatever be the absolute lengths of the sides, the *angles* are the same; for, their homologous sides being proportional, the triangles are similar, and hence mutually equiangular.

(4.) If the *ratios* change, the angles change; for, if the angles do not change, the triangles will be mutually equiangular, and hence the ratios of their sides will be equal, which is contrary to the hypothesis. Hence, *the ratios of the sides determine the angles, and conversely;* that is, determinate values being assigned to one, determinate values may be found for the other. It is by means of this reciprocal property that the solution of triangles is accomplished.

2. Since every oblique triangle may be resolved into two right ones by a perpendicular from any vertex to the opposite side, the solution of all triangles is readily based upon that of right triangles.

Let us, therefore, further consider the right triangles of the annexed diagram, having the same acute angle A.



Since these triangles are similar, we have

$$\frac{AB}{AC}=\frac{AB'}{AC'}=\frac{AB''}{AC''};$$

$$\frac{AB}{BC}=\frac{AB'}{B'C'}=\frac{AB''}{B''C''};$$

$$\frac{BC}{AC}=\frac{B'C'}{AC'}=\frac{B''C''}{AC''}.$$

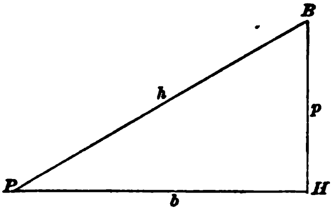
That is, the ratios $AB : BC : AC$ are the same as the corresponding ratios $AB' : B'C' : AC'$, etc.

Hence it appears that in all right triangles having the same acute angle the *ratios* of the sides have constant values; and these values, being independent of the absolute magnitude of the sides, depend, as before observed, solely upon the angle.

If, then, the numerical value of any one of these ratios be given, we shall know the angle which corresponds to it; and, conversely, if the angle be given, we shall know the corresponding ratio.

§ 3. THE TRIGONOMETRIC FUNCTIONS.

These ratios, being so related to the angle that either determines the other, are called the **Trigonometric Functions** of the angle. They have received the special names of *sine, cosine; tangent, cotangent; secant, cosecant; versed sine, coversed sine;* and these are usually written in the abbreviated forms of *sin, cos, tan, cot, sec, csc, vers, cvs.* The functions are distinguished by the following definitions:—



1. *The* **Sine** *of either acute angle is the quotient of the opposite side divided by the hypothenuse:*

Thus,

$$\sin P = \frac{p}{h}, \quad \sin B = \frac{b}{h}.$$

2. *The* **Tangent** *of either acute angle is the quotient of the opposite side divided by the adjacent side:*

$$\tan P = \frac{p}{b}, \quad \tan B = \frac{b}{p}.$$

3. *The* **Secant** *of either acute angle is the quotient of the hypothenuse divided by the adjacent side:*

$$\sec P = \frac{h}{b}, \quad \sec B = \frac{h}{p}.$$

4. *The* **Versed Sine** *of either acute angle is the excess of the hypothenuse over the adjacent side divided by the hypothenuse:*

$$\text{vers } P = \frac{h-b}{h} = 1 - \frac{b}{h}, \quad \text{vers } B = \frac{h-p}{h} = 1 - \frac{p}{h}.$$

5. *The* **Cosine** *of either acute angle is the quotient of the adjacent side divided by the hypothenuse:*

$$\cos P = \frac{b}{h}, \quad \cos B = \frac{p}{h}.$$

6. *The* **Cotangent** *of either acute angle is the quotient of the adjacent side divided by the opposite side:*

$$\cot P = \frac{b}{p}, \quad \cot B = \frac{p}{b}.$$

7. *The* **Cosecant** *of either acute angle is the quotient of the hypothenuse divided by the opposite side:*

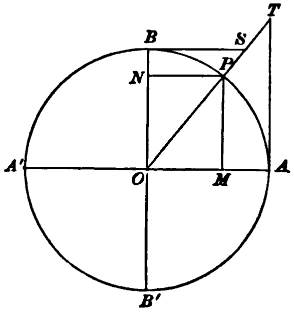
$$\csc P = \frac{h}{p}, \quad \csc B = \frac{h}{b}.$$

8. *The* **Coversed Sine** *of either acute angle is the excess of the hypothenuse over the opposite side, divided by the hypothenuse:*

$$\text{cvs } P = \frac{h-p}{h} = 1 - \frac{p}{h}, \qquad \text{cvs } B = \frac{h-b}{h} = 1 - \frac{b}{h}.$$

NOTE. — These definitions should be thoroughly memorized by the student. They are fundamental to all his work; without them, he can do nothing. We say again, memorize the *definitions*, — these rather than their illustrative equations.

§ 4. REPRESENTATION OF FUNCTIONS BY LINES.



Let OMP be a triangle, right-angled at M. From O as centre with radius OP, describe a circumference. Produce OM to A; draw AT parallel to MP, and meeting OP produced; draw OB perpendicular, NP and BS parallel, to OA; produce AO to A', and BO to B'. AA' wili be the *horizontal*, or *primary*, diameter; BB', the *vertical*, or *secondary*.

In the right triangle OMP,

$$\sin O = \frac{MP}{OP}.$$

$$\cos O = \frac{OM}{OP} = \frac{NP}{OP}.$$

$$\tan O = \frac{MP}{OM} = \frac{AT}{OA}, \text{ since } MP : OM :: AT : OA.$$

$$\cot O = \frac{OM}{MP} = \frac{NP}{ON} = \frac{BS}{OB}.$$

$$\sec O = \frac{OP}{OM} = \frac{OT}{OA}.$$

$$\csc O = \frac{OP}{MP} = \frac{OP}{ON} = \frac{OS}{OB}.$$

$$\text{vers } O = \frac{OP - OM}{OP} = \frac{OA - OM}{OA}.$$

$$\text{cvs } O = \frac{OP - MP}{OP} = \frac{OB - ON}{OB}.$$

If $OA = OP = OB = 1$,

$$\sin\ O = MP = \sin\ \widehat{AP}.$$

$$\cos\ O = NP = \cos\ \widehat{AP}.$$

$$\tan\ O = AT = \tan\ \widehat{AP}.$$

$$\cot\ O = BS = \cot\ \widehat{AP}.$$

$$\sec\ O = OT = \sec\ \widehat{AP}.$$

$$\csc\ O = OS = \csc\ \widehat{AP}.$$

$$\operatorname{vers}\ O = MA = \operatorname{vers}\ \widehat{AP}.$$

$$\operatorname{cvs}\ O = NB = \operatorname{cvs}\ \widehat{AP}.$$

Therefore, in the circle whose radius is unity, *the sine of an arc, or of the angle at the centre measured by that arc, is the perpendicular distance of its terminus* (*P*) *from the primary diameter; the cosine is the perpendicular distance of the terminus* (*P*) *from the secondary diameter; the tangent is the perpendicular to the primary diameter, at the origin* (*A*), *produced until it meets the prolongation of the diameter through the terminus; the versed sine is the perpendicular distance of the origin* (*A*) *from the sine; etc., etc.* In a circle of any other radius than unity, the functions of an arc will be equal to the lines drawn as above, divided by that radius.

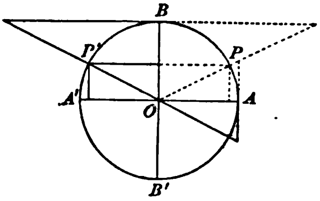
Without limiting the radius to unity, however, these lines are defined, in the Geometrical Method, as the sine, cosine, etc., of the arc or angle to which they belong. It is from this mode of viewing these functions that they have derived their names.

§ 5. EXERCISES.

1. Discuss the functions of an arc extending into the second quadrant, as AP'.

2. Show that the sin of AP equals the sin of AP'.

3. Show that the sin of AP' is less than the tan of AP'.



§ 6. COMPLEMENT OF AN ANGLE.

1. **The Complement of an Angle** is 90° minus the angle.

Thus, the complement of 30° is 60°.

2. It follows that, if the angle is greater than 90°, its complement is *negative*.

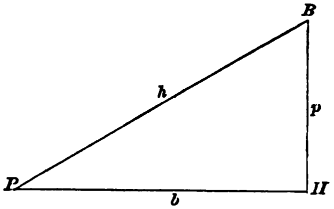
Thus, the complement of 120° is $90° - 120° = -30°$.

3. The acute angles of a right triangle are complements of each other.

4. By definition,

$$\sin P = \frac{p}{h},$$

$$\cos B = \frac{p}{h}.$$



But B and P are complements; hence, *the cosine of an angle equals the sine of its complement, and conversely.*

5. *Similarly, the cotangent, cosecant, and coversed sine of an angle are respectively the tangent, secant, and versed sine of the complement of the angle.*

For convenience of reference, we give the following tabular summary: —

§ 7. TABLE I.

$$(1.)\ \left\{\begin{aligned}\sin P &= \frac{p}{h};\\ \sin B &= \frac{b}{h}.\end{aligned}\right\} \qquad (5.)\ \left\{\begin{aligned}\sec P &= \frac{h}{b};\\ \sec B &= \frac{h}{p}.\end{aligned}\right\}$$

$$(2.)\ \left\{\begin{aligned}\cos B &= \frac{p}{h};\\ \cos P &= \frac{b}{h}.\end{aligned}\right\} \qquad (6.)\ \left\{\begin{aligned}\csc B &= \frac{h}{b};\\ \csc P &= \frac{h}{p}.\end{aligned}\right\}$$

$$(3.)\ \left\{\begin{aligned}\tan P &= \frac{p}{b};\\ \tan B &= \frac{b}{p}.\end{aligned}\right\} \qquad (7.)\ \left\{\begin{aligned}\text{vers } P &= 1 - \cos P;^{1}\\ \text{vers } B &= 1 - \cos B.\end{aligned}\right\}$$

$$(4.)\ \left\{\begin{aligned}\cot B &= \frac{p}{b};\\ \cot P &= \frac{b}{p}.\end{aligned}\right\} \qquad (8.)\ \left\{\begin{aligned}\text{cvs } B &= 1 - \sin B;\\ \text{cvs } P &= 1 - \sin P.\end{aligned}\right\}$$

§ 8. TABLE OF NATURAL FUNCTIONS.

A Table of Natural Functions is a table containing the numerical values of the sine, cosine, tangent, etc., for each given value of the angle, radius being equal to unity. Thus,

$$\begin{aligned}\sin 25^\circ\, 11' &= 0.42552.\\ \cos 41^\circ\, 12' &= 0.75241.\\ \sin 44^\circ\, 36'\, 18'' &= 0.70221.\end{aligned}$$

---

[1] $1 - \frac{b}{h} = 1 - \cos P.$

The table gives, to five decimal places, the natural sine, cosine, tangent, and cotangent of each minute of the quadrant from 0° to 1°, and from 89° to 90°, and the same functions for each ten minutes of the rest of the quadrant.

PROBLEM I.

To find the natural functions of an angle.

For the functions of angles less than 45°, find the degrees and minutes in the left-hand column; for functions of angles greater than 45°, find the degrees and minutes in the right-hand column, the functions being found in the central columns. Functions not given in the table may be found by interpolation.

(1.) Find the nat. sin of $9° 42'$.

$$\text{nat. sin } 9° 40' = 0.1679.$$

$2' = \frac{2}{10}$ of $10'$. The difference between the nat. sin of $9° 40'$ and that of $9° 50' = 0.0029$. $\frac{2}{10}$ of $0.0029 = 0.0006$. $0.1679 + 0.0006 = 0.1685$.

$$\therefore \text{nat. sin of } 9° 42' = 0.1685.$$

(2.) Find the nat. sin of $44° 36' 18''$.

$$\text{nat. sin } 44° 30' = 0.7009.$$

$6' 18'' = \frac{63}{100}$ of $10'$. The difference between the nat. sin of $44° 30'$ and $44° 40' = 0.0021$. $\frac{63}{100}$ of $0.0021 = 0.0013$. $0.7009 + 0.0013 = 0.7022$.

$$\therefore \text{nat. sin } 44° 36' 18'' = 0.7022.$$

§ 9. EXERCISES.

1. Find the natural sine of 35° 25′.
2. Find the natural cosine of 75° 45′.
3. Find the natural tangent of 48° 36′.
4. Find the natural cotangent of 81° 43′.
5. Find the natural sine of 63° 24′ 16″.
6. Find the natural cosine of 48° 36′ 44″.
7. Find the natural tangent of 78° 22′ 36″.
8. Find the natural cotangent of 88° 42′ 14″.

PROBLEM II.

To find the angle corresponding to a given function.

a. When the given function is found exactly in the Table.

If the name of the function is at *the top* of the column in which it is found, the degrees and minutes are *at the left;* if the name of the function is at *the bottom* of the column, the degrees and minutes are *at the right.*

Thus, the angle corresponding to the nat. sin 0.42525 is 25° 10′; the angle corresponding to the nat. tan 2.09654 is 64° 30′.

b. When the given function is not found exactly in the Table.

Find in the table the function next less than the one given, and set aside the corresponding degrees and minutes. Subtract the number found from the one next larger, also from the given number. The numbers thus obtained will show what fractional part of 10′ is to be used as a correction of the degrees and minutes already set aside from the table. Note carefully, however, that this correction, when obtained, is to be *added* in the case of *sines* and *tangents*, but *subtracted* in the case of *cosines* and *cotangents.*

Thus, let it be required to find the angle whose tangent is 2.00945.

Given number, 2.00945
Next less, 2.00569 = tan of $63^\circ 30'$
Difference, 376

Next greater, 2.02039
Next less, 2.00569
Difference, 1470

$\frac{376}{1470}$ of $10' = 2'\ 34''$, correction.
$63^\circ\ 30' + 2'\ 34'' = 63^\circ\ 32'\ 34''$, the angle required.

It is required to find the angle whose natural cosine is 0.34268.

Given number, 0.34268
Next less, 0.34202 = cos 70°
Difference, 66

Next greater, 0.34475
Next less, 0.34202
Difference, 273

$\frac{66}{273}$ of $10' = 2'\ 25''$, correction.
$70^\circ - 2'\ 25'' = 69^\circ\ 57'\ 35''$, the angle required.

§ 10. EXERCISES.

1. Find the angle whose sine is .88445.
2. Find the angle whose sine is .83271.
3. Find the angle whose tangent is 1.3942.
4. Find the angle whose tangent is .80408.
5. Find the angle whose natural cosine is .10279.
6. Find the angle whose natural cosine is .55085.
7. Find the angle whose natural cotangent is .01483.
8. Find the angle whose natural cotangent is .87643.

§ 11. TABLE OF LOGARITHMIC FUNCTIONS.

A table of natural functions could be used for all purposes of trigonometrical computation; but, to avoid the tedious operations of multiplying and dividing by them, it is preferable to use their *logarithms*, which are arranged in a separate table, and called *logarithmic sine*, *tangent*, etc.

Hence, a **Table of Logarithmic Functions** is a table containing the logarithms of the numerical values of the natural functions.

Thus, the natural sine of 14° 30′ = .25038;
and log .25038 = $\bar{1}.3986$,

which is the *logarithmic* sine.

To avoid the introduction of negative characteristics into the table, each logarithm is increased by 10. Hence, the *tabular* logarithmic sine of 14° 30′ is

$$\bar{1}.3986 + 10 = 9.3986.$$

For the sine and cosine, the *tabular differences* are written in separate columns, to the *right;* for the tangent and cotangent, there is but one column of differences, which is written *between* them.

The columns marked 'Sine' and 'Tangent' at the top are marked 'Cosine' and 'Cotangent' at the bottom; for, from the definitions, the *co*-functions of an angle are the functions of the complement, the angles at the *top* and *left* being the complements of those at the *bottom* and *right.*

§ 12. NOTATION.

The tabular logarithm will be denoted by L. in distinction from *log*, the ordinary logarithm. Let A denote any angle, then

$$\begin{aligned} \text{L. } \sin A &= \log \sin A + 10 \\ \text{L. } \cos A &= \log \cos A + 10 \\ \text{L. } \tan A &= \log \tan A + 10 \\ \text{L. } \cot A &= \log \cot A + 10 \end{aligned}$$

NOTE. — It is of the utmost importance that the student should realize that the numbers printed in this table are the logarithms of the sines, etc., *increased* by 10, and are therefore 10 too *great.* Hence, when, in calculations, a tabular logarithmic function is added, 10 must be *subtracted* from the result; when a tabular logarithmic function is subtracted, 10 must be *added* to the result.

PROBLEM III.

To find the tabular logarithmic function of an angle expressed in degrees and minutes.

If the angle is less than 45°, look for the degrees and minutes in the *left*-hand column. Directly opposite to the given number of minutes, and under 'Sine,' 'Cosine,' 'Tangent,' or 'Cotangent,' as the case may be, will be found the logarithm required. Thus,

$$\text{L. sin } 25^\circ\ 30' = 9.6340,$$
$$\text{L. cos } 39^\circ\ 40' = 9.8050.$$

If the angle is greater than 45°, look for the degrees and minutes in the *right*-hand column. Directly opposite to the minutes, and in the column designated at the bottom by the given function, will be found the logarithm required. Thus,

$$\text{L. tan } 64^\circ\ 20' = 10.3183,$$
$$\text{L. cot } 88^\circ\ 40' = 8.3669.$$

In case the exact number of minutes is not given in the table, the logarithmic function may be found by using the numbers in the difference (D) column as indicated in the following examples:

(1.) Find L. sin of 39° 27'.

$$\text{L. sin } 39^\circ\ 20' = 9.8020$$
$$D = 15$$
$$\tfrac{7}{10} \text{ of } 15 = 11$$

$9.8020 + 0.0011 = 9.8031$, the function required.

(2.) Find L. sin of 48° 26' 42''.

$$\text{L. sin } 48^\circ\ 20' = 9.8733$$
$$\text{Tabular difference (D) for } 10' = 12$$
$$6'\ 42'' = \tfrac{402}{600} \text{ of } 10'$$
$$\tfrac{402}{600} \text{ of } 12 = 8$$

$9.8733 + 0.0008 = 9.8741$, the function required.

§ 13. EXERCISES.

1. Find L. sin 43° 28′.
2. Find L. cos 8° 36′.
3. Find L. tan 56° 33′.
4. Find L. cot 78° 22′.
5. Find L. sin 18° 32′ 19″.
6. Find L. cos 66° 54′ 56″.
7. Find L. tan 19° 11′ 13″.
8. Find L. cot 78° 39′ 49″.

PROBLEM IV.

To find the angle corresponding to any tabular logarithmic function.

Reverse the preceding rules.

If the given logarithm is not found exactly, find the *next less*, and set aside the corresponding degrees and minutes. Subtract the logarithm found from the given logarithm, and divide the remainder by the corresponding tabular difference; for the sine and tangent, *add* the quotient to the degrees and minutes set aside; for the cosine and cotangent, *subtract* it.

Thus, let it be required to find the angle whose L. sine is 9.9824.

$$\begin{array}{rl} \text{Given L. sin} = & 9.9824 \\ \text{Sin } 73^\circ\ 40' = \text{next } \textit{less} = & \underline{9.9821} \\ \text{Difference} = & 3 \\ \text{D. } 10' = 4 \text{ and } \frac{3}{4} \text{ of } 10' = & 7'\ 30'' \end{array}$$

$$\therefore \text{ required angle} = 73^\circ\ 40' + 7'\ 30'' = 73^\circ\ 47'\ 30''$$

Let it be required to find the angle whose L. cosine is 9.772561.

$$\begin{aligned} \text{Given L. cos} &= 9.7726 \\ \text{Next } \textit{less} &= \underline{9.7710} \\ \text{Difference} &= \quad\ 16 \end{aligned}$$

$$\text{D. } 10' = 17 \text{ and } \tfrac{16}{17} \text{ of } 10' = 9'\ 25''$$

$$\therefore \text{ required angle} = 53°\ 50' - 9'\ 25'' = 53°\ 40'\ 35''$$

§ 14. EXERCISES.

1. Find the angle whose L. sine is 9.7809.
2. Find the angle whose L. sine is 9.9478.
3. Find the angle whose L. cosine is 9.9166.
4. Find the angle whose L. cosine is 9.7643.
5. Find the angle whose L. tangent is 9.9721.
6. Find the angle whose L. tangent is 10.5310.
7. Find the angle whose L. cotangent is 9.9830.
8. Find the angle whose L. cotangent is 10.1157.
9. Find the L. sine of 6°.

PROBLEM V.

Given any tabular logarithmic function, to find the corresponding natural function.

By definition,

$$\text{L. } \sin A = \log \sin A + 10.$$
$$\therefore \log \sin A = \text{L. } \sin A - 10;$$

Hence, *subtract* 10 *from the given function, and find the number corresponding to the resulting logarithm.*

From the formulas of Table I we get

§ 15. TABLE II.

(1.) $\left\{\begin{aligned} p &= h \sin P; \\ b &= h \sin B. \end{aligned}\right\}$ (2.) $\left\{\begin{aligned} p &= h \cos B; \\ b &= h \cos P. \end{aligned}\right\}$

(3.) $\left\{\begin{aligned} p &= b \tan P; \\ b &= p \tan B. \end{aligned}\right\}$ (4.) $\left\{\begin{aligned} p &= b \cot B; \\ b &= p \cot P. \end{aligned}\right\}$

(5.) $\left\{\begin{aligned} p &= \frac{h}{\sec B}; \\ b &= \frac{h}{\sec P}. \end{aligned}\right\}$ (6.) $\left\{\begin{aligned} p &= \frac{h}{\csc P}; \\ b &= \frac{h}{\csc B}. \end{aligned}\right\}$

Applying logarithms to the formulas of Tables II and I, we get

§ 16. TABLE III.

(1.) $\left\{\begin{aligned} \log p &= \log h + \text{L.} \sin P - 10; \\ \log b &= \log h + \text{L.} \sin B - 10. \end{aligned}\right\}$

(2.) $\left\{\begin{aligned} \log p &= \log h + \text{L.} \cos B - 10; \\ \log b &= \log h + \text{L.} \cos P - 10. \end{aligned}\right\}$

(3.) $\left\{\begin{aligned} \log p &= \log b + \text{L.} \tan P - 10; \\ \log b &= \log p + \text{L.} \tan B - 10. \end{aligned}\right\}$

(4.) $\left\{\begin{aligned} \log p &= \log b + \text{L.} \cot B - 10; \\ \log b &= \log p + \text{L.} \cot P - 10. \end{aligned}\right\}$

(5.) $$\left\{\begin{array}{l} \text{L.}\sin P = 10 + \log p - \log h = \log p + a.c.\log h; \\ \text{L.}\sin B = 10 + \log b - \log h = \log b + a.c.\log h. \end{array}\right\}$$

(6.) $$\left\{\begin{array}{l} \text{L.}\cos P = 10 + \log b - \log h = \log b + a.c.\log h; \\ \text{L.}\cos B = 10 + \log p - \log h = \log p + a.c.\log h. \end{array}\right\}$$

(7.) $$\left\{\begin{array}{l} \text{L.}\tan P = 10 + \log p - \log b = \log p + a.c.\log h; \\ \text{L.}\tan B = 10 + \log b - \log p = \log b + a.c.\log p. \end{array}\right\}$$

(8.) $$\left\{\begin{array}{l} \text{L.}\cot P = 10 + \log b - \log p = \log b + a.c.\log p; \\ \text{L.}\cot B = 10 + \log p - \log b = \log p + a.c.\log b. \end{array}\right\}$$

NOTE. — Let the student now state as *principles* the formulas of Table II. They should be considered and *memorized* as principles rather than as mere formulas, especially (1), (2), (3), and (4). Thus:

Either side adjacent to the right angle is equal to the product of the other side by the tangent of the opposite angle (3).

§ 17. EXERCISES.

1. Find the values of the functions of P if,

(1.) $\left\{\begin{array}{l} p = 3 \\ b = 4 \\ h = 5. \end{array}\right\}$ (2.) $\left\{\begin{array}{l} p = 5 \\ b = 12 \\ h = 13. \end{array}\right\}$

2. Compute the functions of P and B when

(1.) $p = 24$, $b = 143$.
(2.) $b = 9.5$, $h = 19.3$.

3. Compute the functions of P when,

(1.) $p = 2b$, (2.) $p + b = \frac{5}{4}h$.

4. Find p, if $\sin P = \frac{4}{5}$, and $h = 20.5$.

5. Find b, if $\cot P = 4$, and $p = 17$.

6. Construct the angle x if,

(1.) $\tan x = 3$, (2.) $\cos x = \frac{1}{2}$, (3.) $\sin x = 2\cos x$.

§ 18. RIGHT TRIANGLES.

Every plane triangle has six parts, — three sides and three angles.

If three of these parts are given, *one being a side*, the remaining parts can be computed by the principles hitherto established. Three angles alone are insufficient to determine a triangle, since, as we have seen, an infinite number of triangles may have the same three angles. Hence the provision, that at least *one* of the given parts be a *side*.

The various cases in the solution of right triangles are as follows:

1. *Given the hypothenuse and one acute angle.*
2. *Given the hypothenuse and one of the sides adjacent to the right angle.*
3. *Given one side adjacent to the right angle, and one acute angle.*
4. *Given the two sides adjacent to the right angle.*

§ 19. CASE I.

Given the hypothenuse and one acute angle, to find the remaining parts.

$$\text{Given } \left\{ \begin{aligned} h &= 7234.5; \\ P &= 33^\circ\ 19'. \end{aligned} \right\}$$

$$\text{Required } \left\{ \begin{array}{l} B, \\ p, \\ b. \end{array} \right.$$

$$B = 90^\circ - P = 90^\circ - 33^\circ\ 19' = 56^\circ\ 41'.$$

Either side adjacent to the right angle is equal to the product of the hypothenuse by the sine of the opposite angle.

Table II, (1).

$$\begin{aligned}\therefore\ p &= h \sin P.\\ \log p &= \log h + \text{L. } \sin P - 10.\\ &= \log 7234.5 + \text{L. } \sin 33^\circ\ 19' - 10.\end{aligned}$$

$$\begin{aligned}\log 7234.5 &= 3.8594;\\ \text{L. } \sin 33^\circ\ 19' &= \underline{9.7398.}\\ \therefore\ \log p &= 3.5992.\\ \therefore\ p &= 3973.636.\end{aligned}$$

$$\begin{aligned}\log b &= \log h + \text{L. } \sin B - 10.\\ &= \log 7234.5 + \text{L. } \sin 56^\circ\ 41' - 10.\end{aligned}$$

$$\begin{aligned}\log 7234.5 &= 3.8594;\\ \text{L. } \sin 56^\circ\ 41' &= \underline{9.9220.}\\ \therefore\ \log b &= 3.7814.\\ \therefore\ b &= 6045.714.\end{aligned}$$

§ 20. EXERCISES.

1. Given $\left\{\begin{aligned} h &= 345;\\ P &= 30^\circ\ 12'.\end{aligned}\right\}$ Required $\left\{\begin{aligned} B &= 59^\circ\ 48';\\ p &= 173.56;\\ b &= 298.2.\end{aligned}\right\}$

2. Given $\left\{\begin{aligned} h &= 65.07;\\ B &= 39^\circ\ 38'\ 28''.\end{aligned}\right\}$ Required $\left\{\begin{aligned} P &= 50^\circ\ 21'\ 32'';\\ b &= 41.52;\\ p &= 50.125.\end{aligned}\right\}$

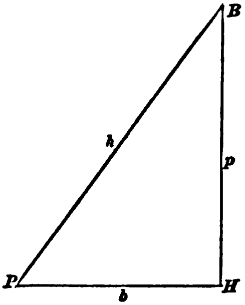
3. Given $\left\{\begin{aligned} h &= 24856;\\ P &= 18^\circ\ 46'\ 14''.\end{aligned}\right\}$ Required $\left\{\begin{aligned} B &= 71^\circ\ 13'\ 46'';\\ p &= 8000;\\ b &= 23538.888.\end{aligned}\right\}$

§ 21. CASE II.

Given the hypothenuse and one of the sides adjacent to the right angle, to find the remaining parts.

$$\text{Given}\begin{cases} h = 1005.4; \\ p = 724.5. \end{cases}$$

$$\text{Required}\begin{cases} P, \\ B, \\ b. \end{cases}$$



The sine of either acute angle is the quotient of the opposite side divided by the hypothenuse . . . § 3, *Def.* 1.

$$\therefore \sin P = \frac{p}{h}.$$

$$\text{L. }\sin P = \log p + a.\,c.\,\log h.$$

$$= \log 724.5 + a.c.\log 1005.4 \quad \text{. . Table III, (5).}$$

$$\log 724.5 = 2.8600;$$

$$a.\,c.\,\log 1005.4 = 6.9977.$$

$$\therefore \text{L. }\sin P = 9.8577.$$

$$\therefore P = 46^\circ\ 6'\ 9''$$

$$B = 90^\circ - P = 90^\circ - 46^\circ\ 6'\ 9'' = 43^\circ\ 53'\ 51''.$$

$$b = h\cos P \quad \text{. . . . . Table II, (2).}$$

$$\log b = \log h + \text{L. }\cos P - 10.$$

$$= \log 724.5 + \text{L. }\cos 46^\circ\ 6'\ 9'' - 10.$$

$$\begin{aligned} \log 724.5 &= 2.8600 \\ \text{L. cos } 46^\circ\ 6'\ 9'' &= \underline{9.8410} \\ \therefore \log b &= 2.7010 \\ \therefore \quad b &= 502.375 \end{aligned}$$

§ 22. EXERCISES.

1. Given $\left\{\begin{array}{l} h = 365; \\ p = 86. \end{array}\right\}$ Required $\left\{\begin{array}{l} P = 13^\circ\ 37\ 42''; \\ B = 76^\circ\ 22'\ 18''; \\ b = 354.724. \end{array}\right\}$

2. Given $\left\{\begin{array}{l} h = 826.725; \\ b = 125. \end{array}\right\}$ Required $\left\{\begin{array}{l} B = 8^\circ\ 41'\ 40''; \\ P = 81^\circ\ 18'\ 20''; \\ p = 817.224. \end{array}\right\}$

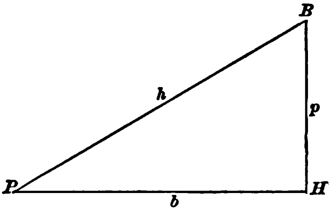
3. Given $\left\{\begin{array}{l} h = .897 \\ p = .00086. \end{array}\right\}$ Required $\left\{\begin{array}{l} P = 3'\ 20''; \\ B = 89^\circ\ 56'\ 40''; \\ b = .89699. \end{array}\right\}$

§ 23. CASE III.

Given one side adjacent to the right angle and one acute angle, to find the remaining parts.

Given $\left\{\begin{array}{l} p = 7643.5; \\ P = 37^\circ\ 18'. \end{array}\right\}$

Required $\left\{\begin{array}{l} B, \\ b, \\ h. \end{array}\right.$



$$B = 90^\circ - P = 90^\circ - 37^\circ\ 18' = 52^\circ\ 42'.$$

Either side adjacent to the right angle is equal to the cotangent of the adjacent acute angle, multiplied by the other side . . . . Table II, (4).

$$\begin{aligned}
\therefore b &= p \cot P. \\
\log b &= \log p + \text{L.} \cot P - 10. \\
&= \log 7643.5 + \text{L.} \cot 37^\circ\ 18' - 10. \\
\therefore b &= 10032.5.
\end{aligned}$$

Either side adjacent to the right angle is equal to the cosine of the adjacent acute angle, multiplied by the hypothenuse.

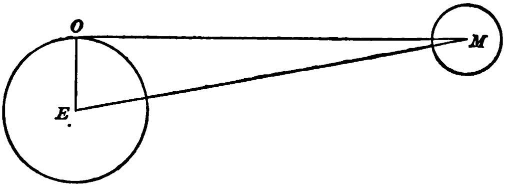
$$\begin{aligned}
\therefore b &= h \cos P; \\
\therefore h &= \frac{b}{\cos P}. \\
\log h &= \log b - \log \cos P \\
&= \log b - \text{L.} \cos P + 10 \\
&= \log b + a.\,c.\ \text{L.} \cos P \\
&= \log 10032.5 + a.\,c.\ \text{L.} \cos 37^\circ\ 18'. \\
\therefore h &= 126133.
\end{aligned}$$

§ 24. EXERCISES.

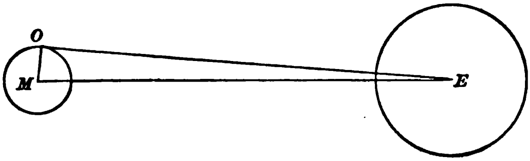
1. Given $\left\{\begin{array}{l} b = 75.63\,; \\ P = 40^\circ\ 44'\ 19''. \end{array}\right\}$ Required $\left\{\begin{array}{l} B = 49^\circ\ 15'\ 41''; \\ p = 65.114\,; \\ h = 99.8. \end{array}\right\}$

2. Given $\left\{\begin{array}{l} EO = 3963.35 = \text{earth's radius}\,; \\ EMO = 57'\ 2.3'' = \text{moon's horizontal parallax.} \end{array}\right\}$

Required $EM = 238888.8$, the distance of the moon from the earth.



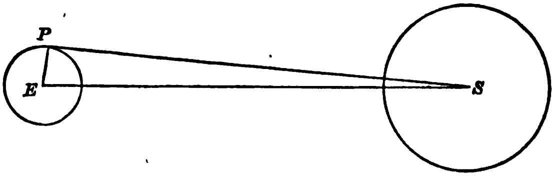
3. Given $\left\{\begin{array}{ll} EM &= 238888.8; \\ OEM &= 15' \; 40''. \end{array}\right\}$ Required the diameter of the moon.



4. Spheres being proportional to the cubes of their diameters, what fraction of the earth's volume is that of the moon?

5. Given $\left\{\begin{array}{ll} EP &= 3963.35 = \text{earth's radius}; \\ ESP &= 8.9'' = \text{sun's horizontal parallax.} \end{array}\right\}$

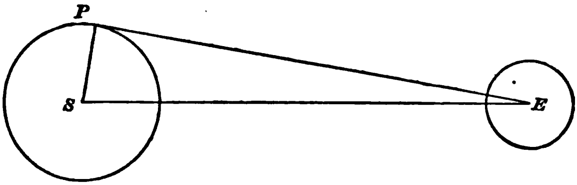
Required ES, the sun's distance from the earth.



$$\left[8.9'' = 1' \times \frac{8.9}{60}; \; \therefore \; \sin 8.9'' = \sin 1' \times \frac{8.9}{60}; \right.$$

$$\left. \therefore \; \text{L.} \sin 8.9'' = \text{L.} \sin 1' + \log 8.9 + a.\,c.\, \log 60 - 10. \right]$$

6. Given $\left\{\begin{array}{ll} ES &= 91854000; \\ SEP &= 16' \; 1.8''. \end{array}\right\}$ Required the sun's diameter.

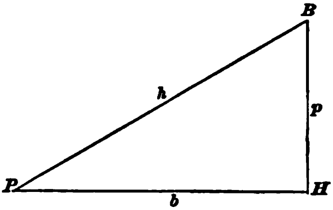


7. How much greater is the volume of the sun than that of the earth?

§ 25. CASE IV.

Given the two sides adjacent to the right angle, to find the remaining parts.

$$\text{Given}\left\{\begin{aligned} b &= 3678;\\ p &= 2876.\end{aligned}\right\}$$



$$\text{Required}\left\{\begin{aligned} B &= 51^\circ\ 58'\ 35'';\\ P &= 38^\circ\ \ 1'\ 25'';\\ h &= 4668.935.\end{aligned}\right\}$$

The tangent of either acute angle is the quotient of the opposite side divided by the adjacent side.

$$\therefore \tan B = \frac{b}{p}.$$

$$\begin{aligned} \text{L. } \tan B &= \log b + a.\,c.\ \log p\\ &= \log 3678 + a.\,c.\ \log 2876.\\ \therefore B &= 51^\circ\ 58'\ 51''.\\ P &= 90^\circ - B = 38^\circ\ 1'\ 9''.\end{aligned}$$

Either side adjacent to the right angle is equal to the product of the hypothenuse by the sine of the opposite angle . . . . Table II, (1).

$$\therefore b = h \sin B;$$

$$\therefore h = \frac{b}{\sin B}.$$

$$\begin{aligned}\log h &= \log b - \log \sin B\\ &= \log b - \text{L. } \sin B + 10\\ &= \log b + a.\,c.\ \text{L. } \sin B\\ &= \log 3678 + a.\,c.\ \text{L. } \sin 51^\circ\, 58'\, 51''.\\ \therefore h &= 4668.94.\end{aligned}$$

§ 26. EXERCISES.

1. Given $\begin{Bmatrix} p = 364.8\,; \\ b = 297.645. \end{Bmatrix}$ Required $\begin{Bmatrix} P = 50^\circ\, 47'\, 12''\,; \\ B = 39^\circ\, 12'\, 48''\,; \\ h = 470.82. \end{Bmatrix}$

2. Given $\begin{Bmatrix} p = 4.876\,; \\ b = 3.006. \end{Bmatrix}$ Required $\begin{Bmatrix} P = 58^\circ\, 20'\, 48''\,; \\ B = 31^\circ\, 39'\, 12''\,; \\ h = 5.728132. \end{Bmatrix}$

3. Given $\begin{Bmatrix} p = .87\,; \\ b = .004. \end{Bmatrix}$ Required $\begin{Bmatrix} P = 89^\circ\, 43'\, 24''\,; \\ B = 16'\, 36''\,; \\ h = .87001. \end{Bmatrix}$

4. Given the base = 469.34, and the adjacent acute angle = 51° 26′ 17″; to find the remaining parts.

5. Given the base = 63.4, and the hypothenuse = 85.72; to find the remaining parts.

6. In a right triangle whose sides are 3, 4, and 5, what are the values of the trigonometrical functions of the angles? (Consult definitions.)

7. *Solve the following by the use of natural numbers:*

(1.) $h = 150$, and $P = 30^\circ$.
(2.) $p = 125$, and $B = 22\frac{1}{2}^\circ$.
(3.) $p = 124.6$, and $P = 64^\circ\ 20'$.
(4.) $p = 141$, and $b = 193$.
(5.) $h = 656.15$, and $B = 55^\circ\ 30'\ 17''$.
(6.) The side of a rhombus is 48 feet long, and one of its angles is 68°; find the length of its diagonals.

§ 27. TABULAR SUMMARY.

Let h = the hypothenuse, b the base, B the acute angle opposite, p the perpendicular, and P the acute angle at the base.

GIVEN.	REQUIRED.	FORMULAS.
p, b	h, P, B	$\tan P = \frac{p}{b}$; $B = 90° - P$; $h = \frac{p}{\sin P}$.
p, h	b, P, B	$\sin P = \frac{p}{h}$; $B = 90° - P$; $b = h \sin B$.
p, P	b, h, B	$B = 90° - P$; $b = p \tan B$; $h = \frac{b}{\sin B}$.
b, P	p, h, B	$B = 90° - P$; $p = b \tan P$; $h = \frac{p}{\sin P}$.
h, P	p, b, B	$B = 90° - P$; $p = h \sin P$; $b = h \sin B$.
p, B	b, h, P	$P = 90° - B$; $b = p \tan B$; $h = \frac{p}{\sin P}$.
b, B	p, h, P	$P = 90° - B$; $p = b \tan P$; $h = \frac{b}{\sin B}$.
h, B	p, b, P	$P = 90° - B$; $p = h \sin P$; $b = h \sin B$.

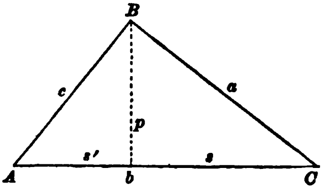
§ 28. OBLIQUE TRIANGLES.

The cases of oblique triangles are reducible to four:

1. *Given the three sides.*
2. *Given one side and two angles.*
3. *Given two sides and their included angle.*
4. *Given two sides and an angle opposite one of them.*

§ 29. CASE I.

Given the three sides, to find the angles.



Let ABC be any plane triangle of which AC, assumed as the base, is the longest side; and let fall a perpendicular from the vertex to the base. Then,

$$(1) \qquad a^2 = s^2 + p^2;$$

$$\text{and } (2) \qquad c^2 = s'^2 + p^2.$$

$$(1) - (2) = (3) \qquad a^2 - c^2 = s^2 - s'^2;$$

$$\therefore (4)\ (a + c)\ (a - c) = (s + s')\ (s - s');$$

$$\therefore (5)\ s + s' : a + c :: a - c : s - s'.$$

Hence, *if a perpendicular be drawn from the vertical angle of a triangle to the base, the whole base will be to the sum of the other two sides as the difference of those sides is to the difference of the segments of the base.*

$$(5) \text{ gives } (6) \qquad s - s' = \frac{(a + c)\ (a - c)}{b}.$$

$$(7) \log (s - s') = \log (a + c) + \log (a - c) + \text{a.c.} \log b - 10.$$

When the numbers are small, find the difference of the segments from (6); otherwise, from (7).

Half the difference of the segments added to half the base gives the greater, and subtracted from half the base gives the less; for, let $s - s' = x'$,

and $s + s' = x''$;

then $$s = \frac{x' + x''}{2} = \tfrac{1}{2}x' + \tfrac{1}{2}x''$$

$=$ the half difference $+$ the half base;

and $$s' = \frac{x'' - x'}{2} = \tfrac{1}{2}x'' - \tfrac{1}{2}x'$$

$=$ the half base $-$ the half difference.

The segments being found, we shall then have two right triangles, in each of which the hypothenuse and the base are known.

$$\text{Given}\begin{Bmatrix} a = 4652; \\ b = 5928.7; \\ c = 3756.3. \end{Bmatrix} \qquad \text{Required}\begin{Bmatrix} A, \\ B, \\ C. \end{Bmatrix}$$

$$s - s' = \frac{(a+c)\,(a-c)}{b} = \frac{(8408.3)\,(895.7)}{5928.7}.$$

$$\therefore s - s' = 1270.315.$$

$$s + s' = 5928.7.$$

$$\therefore s = 3599.51,$$

$$\text{and } s' = 2329.19.$$

The cosine of either acute angle is equal to the adjacent side divided by the hypothenuse.

$$\therefore \cos A = \frac{s'}{c}.$$

$$\text{L. } \cos A = \log s' + a.\,c.\,\log c$$

$$= \log 2329.19 + a.\,c.\,\log 3756.3.$$

$$\therefore A = 51^\circ\ 41'\ 15''.$$

$$\cos\ C = \frac{s}{a}.$$

$$\begin{aligned} \text{L. } \cos\ C &= \log\ s + a.\ c.\ \log\ a \\ &= \log\ 3599.51 + a.\ c.\ \log\ 4652. \\ \therefore\ C &= 39^\circ\ 19'\ 4''. \\ B &= 180^\circ - (A + C) = 88^\circ\ 59'\ 41''. \end{aligned}$$

§ 30. EXERCISES.

1. Given $\left\{\begin{array}{l} a = 14; \\ b = 16; \\ c = 10. \end{array}\right\}$ Required $\left\{\begin{array}{l} A = 60^\circ; \\ B = 81^\circ\ 48'; \\ C = 38^\circ\ 12'. \end{array}\right\}$

2. Given $\left\{\begin{array}{l} a = 5124; \\ b = 4963; \\ c = 5621. \end{array}\right\}$ Required $\left\{\begin{array}{l} A = 57^\circ\ 29'\ 53''; \\ B = 54^\circ\ 47'\ 13''; \\ C = 67^\circ\ 42'\ 54''. \end{array}\right\}$

3. Given $\left\{\begin{array}{l} a = 72.81; \\ b = 61.47; \\ c = 58.38. \end{array}\right\}$ Required $\left\{\begin{array}{l} A = 74^\circ\ 46'\ 13''; \\ B = 54^\circ\ 33'\ 20''; \\ C = 50^\circ\ 40'\ 27''. \end{array}\right\}$

4. Given $\left\{\begin{array}{l} a = 353.25; \\ b = 275.5; \\ c = 372. \end{array}\right\}$ Required $\left\{\begin{array}{l} A = 64^\circ\ \ 6'\ 56''; \\ B = 44^\circ\ 33'\ 20''; \\ C = 71^\circ\ 19'\ 44''. \end{array}\right\}$

§ 31. CASE II.

Given one side and two angles, to find the remaining parts.

THEOREM.

The sine of an angle is equal to the sine of its supplement.[1]

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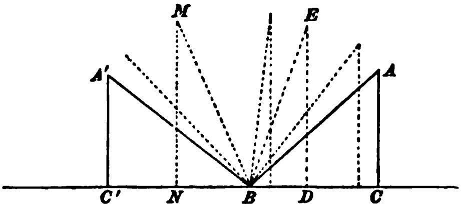
[1] The remainder obtained by subtracting the angle from 180°.



Conceive AB to revolve about B from the position BC to $A'B$, where angle $A'BC' = ABC$.

In each position of AB, let perpendiculars be dropped, and let x denote the variable angle which the revolving line successively makes with the fixed line CC', from its first position to its last.

Then, by extending the definition of Art. 3, the sine of x for *any* position of AB is the perpendicular divided by the hypothenuse; that is,

for the position AB, $\sin x = \frac{CA}{BA} = \sin ABC$;

for the position EB, $\sin x = \frac{DE}{BE} = \sin EBC$;

for the position MB, $\sin x = \frac{NM}{BM} = \sin MBC$;

for the position $A'B$, $\sin x = \frac{C'A'}{BA'} = \sin A'BC$.

But, $\frac{C'A'}{BA'} = \sin A'BC'$;

$\therefore$ (1) $\sin A'BC' = \sin A'BC$.

But, $A'BC' = ABC$, and $\therefore A'BC = ABC'$;[1]

$\therefore$ (2) $\sin ABC = \sin ABC'$;

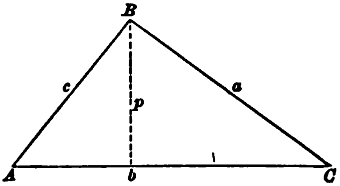
that is, $\sin ABC = \sin (180° - ABC)$.

[1] $A'BC = 180° - A'BC' = 180° - ABC = ABC'$.

THEOREM.

In any plane triangle, the sines of the angles are proportional to the opposite sides.

Let ABC be any plane triangle, and p a perpendicular from one of its vertices to the opposite side, dividing the given triangle into two right ones. Then,

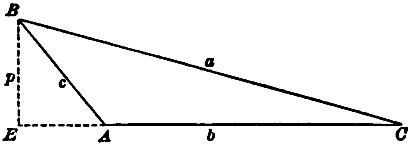


$$p = c \sin A, \text{ and } p = a \sin C;$$

$$\therefore c \sin A = a \sin C;$$

$$\therefore \sin A : \sin C :: a : c.$$

If the perpendicular falls upon the side produced,



$$p = c \sin EAB, \text{ and } p = a \sin C.$$

But EAB is the supplement of A;

$$\therefore \sin EAB = \sin A, \text{ and therefore}$$

$$p = c \sin A;$$

$$\therefore c \sin A = a \sin C;$$

$$\therefore \sin A : \sin C :: a : c.$$

In like manner,

$$\sin A : \sin B :: a : b.$$

Hence, *in any plane triangle, the sines of the angles are proportional to the opposite sides.*

Scholium. — Assume the angles A and C and the side a to be given; then,

$$B = 180^\circ - (A + C).$$

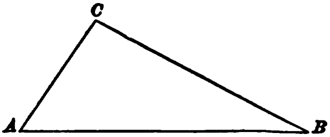
To find the sides, the preceding principle may be stated thus:

The sine of the angle opposite the given side is to the sine of the angle opposite the required side as the given side is to the required side.

Thus, $\sin A : \sin B :: a : b,$

$$\sin A : \sin C :: a : c.$$

Given $\begin{cases} b - 95.4; \\ A = 49^\circ; \\ C = 29^\circ\ 19'. \end{cases}$



Required $\begin{cases} B, \\ a, \\ c. \end{cases}$

$$B = 180^\circ - (A + C) = 180^\circ - 78^\circ\ 19' = 101^\circ\ 41'.$$

$$\sin A : \sin B :: a : b;$$

$$\therefore\quad a = b\,\frac{\sin A}{\sin B}.$$

$$\begin{aligned} \log a &= \log b + \log \sin A - \log \sin B \\ &= \log b + \text{L.} \sin A + a.\,c.\ \text{L.} \sin B - 10 \\ &= \log 95.4 + \text{L.} \sin 49^\circ + a.\,c.\ \text{L.} \sin 101^\circ\ 41' - 10. \end{aligned}$$

$$\therefore\quad a = 73.5.$$

[In finding L. sin 101° 41′, find L. sin (180° − 101° 41′) = L. sin 78° 19′.]

$$\sin A : \sin C :: a : c;$$

$$\therefore \quad c = a\,\frac{\sin C}{\sin A}.$$

$$\log c = \log a + \text{L. } \sin C + a.\,c.\ \text{L. } \sin A - 10$$

$$= \log 73.5 + \text{L. } \sin 29^\circ\, 19' + a.\,c.\ \text{L. } \sin 49^\circ - 10.$$

$$\therefore \quad c = 47.7.$$

§ 32. EXERCISES.

1. Given $\left\{\begin{array}{l} b = 15.75; \\ B = 75^\circ\, 40'\, 44''; \\ C = 65^\circ\, 36'\, 56''. \end{array}\right\}$ Required $\left\{\begin{array}{l} A = 38^\circ\, 42'\, 20''; \\ a = 10.16; \\ c = 14.8. \end{array}\right\}$

2. Given $\left\{\begin{array}{l} a = 6984; \\ A = 36^\circ\, 48'; \\ B = 48^\circ\, 36'. \end{array}\right\}$ Required $\left\{\begin{array}{l} C = 94^\circ\, 36'; \\ b = 8746; \\ c = 11622.5. \end{array}\right\}$

3. Given $\left\{\begin{array}{l} c = 696; \\ A = 81^\circ\, 30'\, 10''; \\ B = 40^\circ\, 30'\, 44''. \end{array}\right\}$ Required $\left\{\begin{array}{l} C = 57^\circ\, 59'\, 6''; \\ a = 811.83; \\ b = 533.25. \end{array}\right\}$

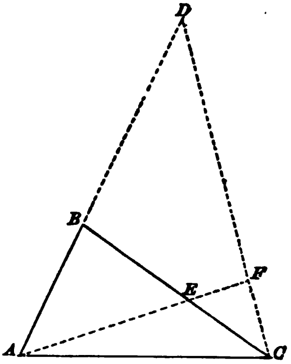
4. Given $\left\{\begin{array}{l} b = 696.75; \\ A = 68^\circ\ \ 4'\, 20''; \\ B = 56^\circ\, 32'\, 35''. \end{array}\right\}$ Required $\left\{\begin{array}{l} C = 55^\circ\, 23'\, 5''; \\ a = 774.833; \\ c = 687.33. \end{array}\right\}$

§ 33. CASE III.

Given two sides and their included angle, to find the remaining parts.

Let ABC be any plane triangle in which the two sides, AB, BC, and their included angle ABC are given.

Produce AB to D, making $BD = BC$. Draw CD.



Take $BE = AB$, and through E draw AF.

Then AD = *sum* of the given sides,

and CE = their *difference*.

$$BAE + AEB = 180^\circ - ABC,$$

$$BAC + ACB = 180^\circ - ABC;$$

$\therefore\ BAE + AEB = BAC + ACB.$ But $BAE = AEB$;

$$\therefore\quad 2\,BAE = BAC + ACB,$$

and $\quad BAE = \dfrac{BAC + ACB}{2}$ = the half *sum*.

$\quad CAF = \dfrac{BAC - ACB}{2}$ = the half *difference*;[1]

for, $\quad CAF = BAC - BAE$

= the *greater* − the half *sum*.

Again, $\quad BDC = BCD$, since $BD = BC$;

and $\quad BAE = AEB = CEF$.

---

[1] From Case I, the greater = the half *sum* + the half *difference*; hence, the half *difference* = the greater − the half *sum*.

Hence, the triangles ADF and CEF have two angles of the one equal to two angles of the other, each to each, and are therefore similar, since their third angles AFD and EFC must be equal. But, if $AFD = EFC$, AF is *perpendicular* to DC.

By similar triangles,

$$AD : EC :: DF : CF :: \frac{DF}{AF} : \frac{CF}{AF}.$$

But $\frac{DF}{AF} = \tan BAE = \tan \frac{1}{2}(A + C)$,

and $\frac{CF}{AF} = \tan CAF = \tan \frac{1}{2}(A - C)$ . . . *Art.* 3, *Def.* 2.

$\therefore CB + AB : CB - AB :: \tan \frac{1}{2}(A + C) : \tan \frac{1}{2}(A - C)$.

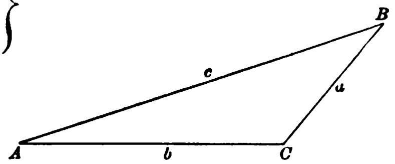
Hence, *in any plane triangle, the sum of any two sides is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference.*

Scholium.—The half difference added to the half sum gives the greater angle, and, subtracted from the half sum, gives the less angle.

Having found the remaining two angles, the third side may be found by Case II.

$$\text{Given} \begin{cases} a = 320; \\ b = 562; \\ C = 128° \ 4'. \end{cases}$$

$$\text{Required} \begin{cases} A, \\ B, \\ c. \end{cases}$$



$$b + a : b - a :: \tan \tfrac{1}{2}(B + A) : \tan \tfrac{1}{2}(B - A).$$

$$\therefore \tan \tfrac{1}{2}(B - A) = \frac{(b - a) \tan \frac{1}{2}(B + A)}{b + a}.$$

$$\begin{aligned}\therefore \text{L.} \tan \tfrac{1}{2}(B - A) &= \log(b - a) + \text{L.} \tan \tfrac{1}{2}(B + A) \\ &\qquad + a.c. \log(b + a) - 10 \\ &= \log 242 + \text{L.} \tan 25^\circ\, 58' \\ &\qquad + a.c. \log 882 - 10.\end{aligned}$$

$$\begin{aligned}\therefore \tfrac{1}{2}(B - A) &= 7^\circ\, 36'\, 42''. \\ B &= \tfrac{1}{2}(B + A) + \tfrac{1}{2}(B - A) \\ &= 25^\circ\, 58' + 7^\circ\, 36'\, 42'' \\ &= 33^\circ\, 34'\, 42''. \\ A &= \tfrac{1}{2}(B + A) - \tfrac{1}{2}(B - A) \\ &= 25^\circ\, 58' - 7^\circ\, 36'\, 42'' \\ &= 18^\circ\, 21'\, 18''\end{aligned}$$

$$\sin A : \sin C :: a : c.$$

$$\therefore c = \frac{a \sin C}{\sin A}.$$

$$\begin{aligned}\log c &= \log a + \text{L.} \sin C + a.c.\ \text{L.} \sin A - 10 \\ &= \log 320 + \text{L.} \sin 51^\circ\, 56' \\ &\qquad + a.c.\ \text{L.} \sin 18^\circ\, 21'\, 18'' - 10. \\ \therefore c &= 799.857.\end{aligned}$$

§ 34. EXERCISES.

1. Given $\left\{\begin{array}{l} a = 3656; \\ b = 2474; \\ C = 66^\circ\, 26'. \end{array}\right\}$ Required $\left\{\begin{array}{l} A = 73^\circ\, 11'\, 28''; \\ B = 40^\circ\, 22'\, 32''; \\ c = 3500. \end{array}\right\}$

2. Given $\left\{\begin{array}{l} a = 148.4; \\ c = 468.3; \\ B = 104^\circ\, 18'. \end{array}\right\}$ Required $\left\{\begin{array}{l} A = 15^\circ\, 53'\, 30''; \\ C = 59^\circ\, 48'\, 30''; \\ b = 525. \end{array}\right\}$

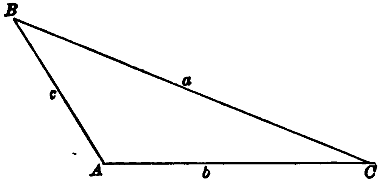
3. Given $\left\{\begin{array}{l} b = 666.666\,; \\ c = 333.4\,; \\ A = 56^\circ\,48'. \end{array}\right\}$ Required $\left\{\begin{array}{l} B = 93^\circ\,14'\,47''\,; \\ C = 29^\circ\,57'\,13''\,; \\ a = 558.74. \end{array}\right\}$

4. Given $\left\{\begin{array}{l} a = 468\,; \\ b = 365\,; \\ C = 82^\circ\,22'\,22''. \end{array}\right\}$ Required $\left\{\begin{array}{l} A = 56^\circ\,51'\,27''\,; \\ B = 40^\circ\,46'\,13''\,; \\ c = 554. \end{array}\right\}$

§ 35. CASE IV.

Given two sides and an angle opposite one of them, to find the remaining parts.

1. Given a, b, and $A > 90^\circ$; required B, C, and c.



Since, in any plane triangle, the sines of the angles are proportional to the opposite sides,

$$\sin A : \sin B :: a : b; \text{ whence}$$

$$\sin B = \frac{b \sin A}{a}.$$

Now this sine belongs to two angles, B and the supplement of B . . . . *Case II, Th.*

How are we to determine which of the two to take?

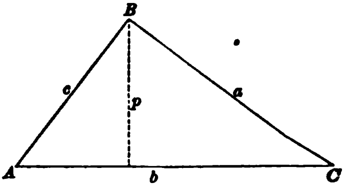
Since the given angle is *obtuse*, the required angle, B, must be *acute;* and in this case, as in any similar case where the given angle is obtuse, the *supplement* is inadmissible.[1]

---

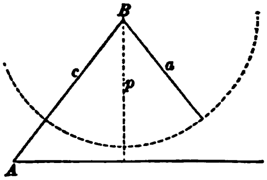
[1] Since the sum of the angles of a triangle equals two right angles.

The 'ambiguous' case, then, can arise only when the given angle is *acute*, and here the several cases depend upon the value of the *side opposite the given angle.*

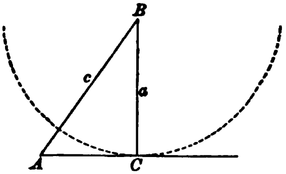
2. Given a, c, and $A < 90°$; required B, C, and b.



(1.) If a were less than p, the problem would be impossible, for the triangle, under such conditions, could not be constructed.[1]



(2.) If $a = p$, the triangle is right-angled at C, and the third angle, B, must be *acute.*



(3.) If $a > p$ and $< c$, there are *two* triangles, both of which fulfil the given conditions.

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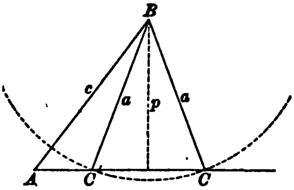
[1] Let the student consult his geometry for the method of constructing a triangle, having given two sides and an angle opposite one of them.



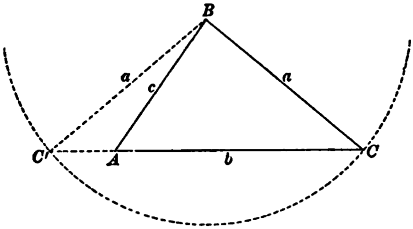
Thus, in both ABC and ABC', $a > p$ and $< c$, and angle $A < 90°$.

Hence, in the formula

$$\sin C = \frac{c \sin A}{a},$$

both angles corresponding to $\sin C$ may, in this case, be taken.

(4.) If $a > c$, there is but one triangle which fulfils the conditions; for, though there are two triangles, the second is excluded, since it does not contain the given *acute* angle A.



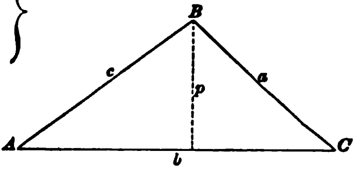
Scholium. — The value of p may be computed from the formula,

$$p = c \sin A.$$

[The ambiguous case of a triangle arises only when the angle is determined by its sine.]

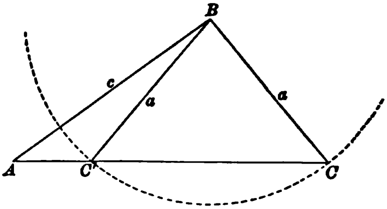
$$\text{Given} \left\{\begin{array}{l} a = 1257.5; \\ c = 1751; \\ A = 31^\circ\ 17'\ 19''. \end{array}\right\}$$

$$\text{Required} \left\{\begin{array}{l} C, \\ B, \\ b. \end{array}\right.$$



$$\begin{aligned} p &= c \sin A. \\ \log p &= \log c + \text{L.} \sin A - 10 \\ &= \log 1751 + \text{L.} \sin 31^\circ\ 17'\ 19'' - 10. \\ \therefore p &= 909.25. \end{aligned}$$

Hence $a > p$ and $< c$, and $A < 90^\circ$; and there are two triangles which will fulfil the given conditions, ABC and ABC'.



$$\sin A : \sin C :: a : c;$$

$$\therefore \sin C = \frac{c \sin A}{a}.$$

$$\begin{aligned} \text{L.} \sin C &= 10 + \log c + \text{L.} \sin A - 10 - \log a \\ &= \log c + \text{L.} \sin A + a.\,c. \log a - 10 \\ &= \log 1751 + \text{L.} \sin 31^\circ\, 17'\, 19'' + a.\,c. \log 1257.5 - 10. \\ \therefore C &= 46^\circ\ 18'\ 20'' \text{ or } 133^\circ\ 41'\ 40''. \\ AC'B &= 133^\circ\ 41'\ 40'', \text{ since} \\ AC'B &= 180^\circ - BC'C = 180^\circ - BCC' = 180^\circ - 46^\circ\, 18'\, 20''. \\ B &= 180^\circ - (A + C) = 180^\circ - 77^\circ\ 35'\ 39'', \text{ or} \\ &\qquad 180^\circ - 164^\circ\ 58'\ 59'' \\ &\quad = 102^\circ\ 24'\ 21'', \text{ or } 15^\circ\ 1'\ 1''. \end{aligned}$$

$$ABC' = 15^\circ\ 1'\ 1'', \text{ since}$$

$$ABC' = 180^\circ - (A + AC'B) = 15^\circ\ 1'\ 1''.$$

To find b, $\sin A : \sin B :: a : b$;

$$\therefore b = \frac{a \sin B}{\sin A}.$$

$$\begin{aligned}\log b &= \log a + \text{L.} \sin B + a.\,c.\ \text{L.} \sin A - 10\\ &= \log 1257.5 + \text{L.} \sin 102^\circ\ 24'\ 21''\\ &\qquad + a.\,c.\ \text{L.} \sin 31^\circ\ 17'\ 19'' - 10.\end{aligned}$$

$$\therefore b = 2365.56.$$

From the second value of B ($15^\circ\ 1'\ 1'' = ABC'$), we find the second value of b (or AC') $= 627.43$.

§ 36. EXERCISES.

1. Given $\left\{\begin{array}{l} a = 120; \\ c = 218; \\ A = 33^\circ\ 12'. \end{array}\right\}$ Required $\left\{\begin{array}{l} B = (?) \\ C = (?) \\ c = (?) \end{array}\right\}$

How many solutions ?

2. Given $\left\{\begin{array}{l} b = 196; \\ c = 125; \\ B = 118^\circ\ 44'\ 48''. \end{array}\right\}$ Required $\left\{\begin{array}{l} A = (?) \\ C = (?) \\ a = (?) \end{array}\right\}$

3. Given $\left\{\begin{array}{l} a = 1506.75; \\ c = 694.8; \\ A = 48^\circ\ 16'. \end{array}\right\}$ Required $\left\{\begin{array}{l} B = (?) \\ C = (?) \\ b = (?) \end{array}\right\}$

Why is the problem impossible ?

4. Given $\left\{\begin{array}{l} a = 40; \\ b = 50; \\ A = 96^\circ\ 22'. \end{array}\right\}$ Required $\left\{\begin{array}{l} B = (?) \\ C = (?) \\ c = (?) \end{array}\right\}$

Why, from the data, but one solution ?

5. Given two sides of a triangle = 28.6 and 18.9, and the angle opposite the first side = 32° 18′; required the remaining parts.

6. Solve the following by the use of natural numbers:

(1) A side = 121, and the adjacent angles = 15°, and 55° 31′.

(2) A side = 14.3, angle opposite = 55°, and the other sides equal.

(3) $a = 11$, $b = 13$, $c = 16$.
Ans. $A = 43° 2' 56''$, $B = 53° 46' 43''$, $C = 83° 10' 21''$.

(4) $B = 49° 30'$, $b = 5$, $c = 6$.
Ans. $A = 64° 38' 54''$, $C = 65° 51' 6''$, $a = 5.942$;
or $A = 16° 21' 6''$, $C = 114° 8' 54''$, $a = 1.851$.

§ 37. DEFINITIONS.

1. The **Horizon** is the apparent junction of the earth and sky.

2. A **Horizontal Plane** is one which is parallel to the horizon. Or it may be defined as a plane tangent to the earth's surface at a given point.

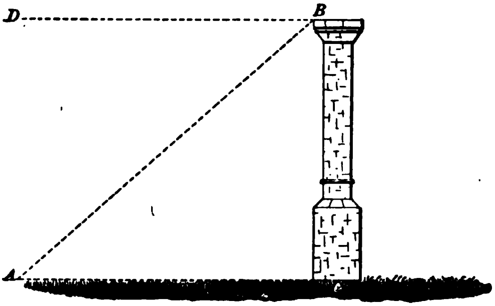
3. A **Vertical Plane** is one which is perpendicular to a horizontal plane.

4. A **Horizontal Line** is one which is parallel to a horizontal plane.

5. A **Vertical Line** is one which is perpendicular to a horizontal plane. A plummet freely suspended and allowed to come to a state of rest assumes a *vertical* position.

6. A **Horizontal Angle** is one the plane of whose sides is horizontal.

7. A **Vertical Angle** is one the plane of whose sides is vertical.



8. An **Angle of Elevation** is a vertical angle having one side horizontal and the other ascending; as *BAC*.

9. An **Angle of Depression** is a vertical angle having one side horizontal and the inclined side descending; as *ABD*.

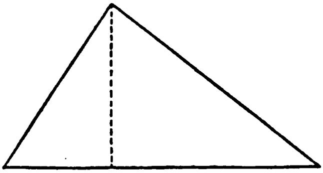
In the latter case, the observer is at *B*; in the former, at *A*.

§ 38. APPLICATIONS.

1. A railroad track 625.75 feet in length, has a uniform grade of 3° 30′; find the vertical rise.

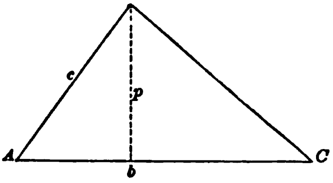
2. Given the vertical rise = 325 feet, and length of track = 9785.5 feet; required the grade.

3. Find the area of a triangular lot whose sides are 30, 40, and 50.

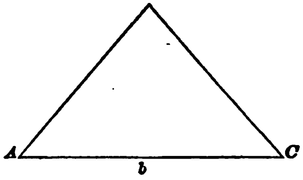


4. Two sides of a triangle are 26 rods and 22.7 rods, and their included angle = 48° 32′ 24″; find the area.

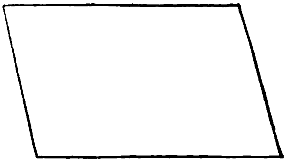
Suggestion. — Find p from A and c. Area $= \frac{1}{2} b p$.



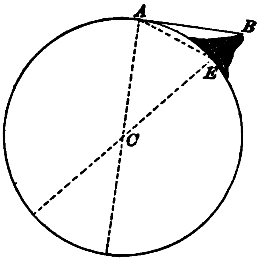
5. Given one side of a triangular field and the two adjacent angles, to find the area.



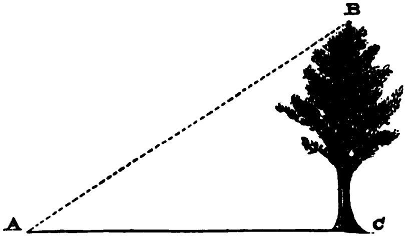
6. A field is in the form of a parallelogram, two of whose adjacent sides are 120 rods and 110 rods, and the included angle = 80° 30′; required the area. (Find the altitude.)



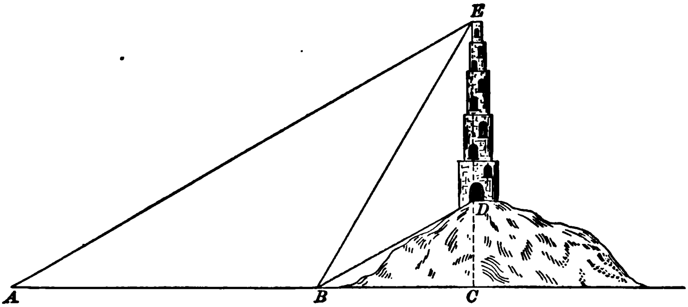
7. Given the height of a mountain = 4 miles, and the angle measured at the top, as between a vertical and a line touching the earth in the horizon, = 87° 25′ 55″; required the surface and volume of the earth, and the greatest distance that can be seen from the summit of the mountain.



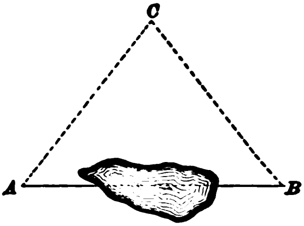
8. Required the height of a tree, *BC*, standing on a horizontal plane, the distance from *C* to *A* being 175 feet, and the angle of elevation, *A*, being 33° 40′.



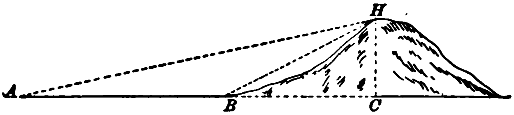
9. Wanting to ascertain the height of a tower standing on a hill, I measured the angle of elevation to the top of the hill, 29°, to the top of the tower, 60°; measuring in a direct line 160 feet, I found the angle of elevation to the top of the tower to be 30°; required the height of the tower.



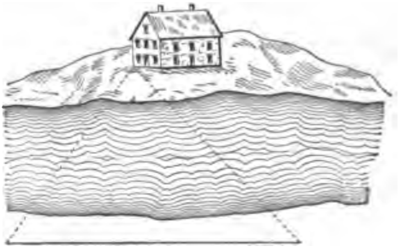
10. It being impracticable to measure from *A* to *B* in a direct line, the distances *AC* and *BC* were measured and found to be 632 rods and 686 rods respectively, and their included angle 74° 15′ 25″; required the distance AB.



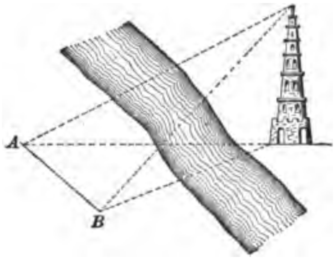
11. Wanting to ascertain the vertical height of a hill, I found the angle of elevation at the base to be 27° 45′; measuring back 875 feet, I found the angle of elevation to be 12° 36′ 48″; required the height of the hill.



12. Being on one side of a river, and wishing to ascertain the distance to a house which stood on the opposite side, I measured 350 yards in a right line by the side of the river, and then found that the two angles at the ends of this line, subtended by the other end and the house, were 57° 20′ 44″ and 48° 38′ 58″; required the distance between each station and the house.

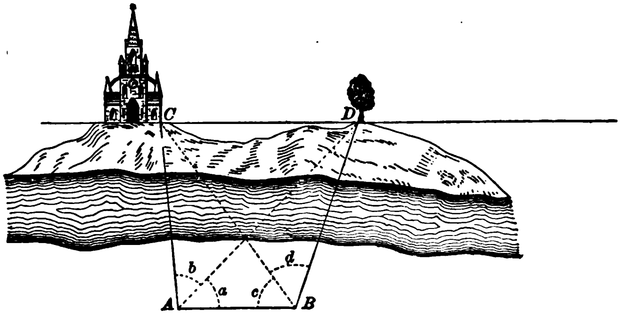


13. Being on one side of a river, and wishing to know the height of a spire on the opposite side, I selected two stations, *A* and *B*, 400 feet apart; the angle of elevation at *B* was 25° 18′, and the horizontal angle at this station between the spire and the first station was 117° 48′ 26″; the horizontal angle at *A* between the spire and the second station was 41° 28′ 15″; required the height of the spire, and the distance from the top of the spire to the first station.



14. Given $\left\{\begin{array}{rl} AB = & 312 \text{ yards}; \\ a = & 46^\circ\ 24'; \\ b = & 49^\circ\ 45'; \\ c = & 54^\circ\ 16'; \\ d = & 55^\circ\ 30'. \end{array}\right\}$

Required *CD*, the distance between two inaccessible objects.



Given.	Required.	Formulas.
$a,\ b,\ c$	$A,\ B,\ C$	$\sin\frac{1}{2}A = \sqrt{\frac{(\frac{1}{2}s-b)(\frac{1}{2}s-c)}{bc}}$; $\sin\frac{1}{2}B = \sqrt{\frac{(\frac{1}{2}s-a)(\frac{1}{2}s-c)}{ac}}$; $C = 180^\circ - (A+B)$.[1]
$b,\ A,\ C$	$B,\ a,\ c$	$B = 180^\circ - (A+C)$; $a = \frac{b \sin A}{\sin B}$; $c = \frac{b \sin C}{\sin B}$.
$a,\ A,\ B$	$C,\ b,\ c$	$C = 180^\circ - (A+B)$; $b = \frac{a \sin B}{\sin A}$; $c = \frac{a \sin C}{\sin A}$.
$c,\ A,\ B$	$C,\ a,\ b$	$C = 180^\circ - (A+B)$; $a = \frac{c \sin A}{\sin C}$; $b = \frac{c \sin B}{\sin C}$.
$b,\ A,\ B$	$C,\ a,\ c$	$C = 180^\circ - (A+B)$; $a = \frac{b \sin A}{\sin B}$; $c = \frac{b \sin C}{\sin B}$.
$a,\ b,\ C$	$A,\ B,\ C$	$\left\{\begin{array}{l}\tan\frac{1}{2}(A-B) = \frac{(a-b)\tan\frac{1}{2}(A+B)}{a+b};\ \textit{the greater angle}\\ = \frac{1}{2}(A+B) + \frac{1}{2}(A-B);\ \textit{the less} = \frac{1}{2}(A+B) - \frac{1}{2}(A-B);\ c = \frac{a \sin C}{\sin A}.\end{array}\right.$
$b,\ c,\ A$	$B,\ C,\ a$	$\left\{\begin{array}{l}\tan\frac{1}{2}(B-C) = \frac{(b-c)\tan\frac{1}{2}(B+C)}{b+c};\ \textit{the greater angle}\\ = \frac{1}{2}(B+C) + \frac{1}{2}(B-C);\ \textit{the less} = \frac{1}{2}(B+C) - \frac{1}{2}(B-C);\ a = \frac{b \sin A}{\sin B}.\end{array}\right.$
$a,\ b,\ A < 90^\circ$	$B,\ C,\ c$	$\sin B = \frac{b \sin A}{a}$; $C = 180^\circ - (A+B)$; $c = \frac{a \sin C}{\sin A}$. Two solutions, when $a < b$, and $a > b \sin A$. One solution when $a = b$, or $a > b$, or $a = b \sin A$. No solution when $a < b \sin A$.
$a,\ b,\ A > 90^\circ$	$B,\ C,\ c$	Two solutions never. No solution when $a < b$, or $a = b$.

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[1] See § 36, Th. II, p. 68, for proof of this formula.

CHAPTER III.

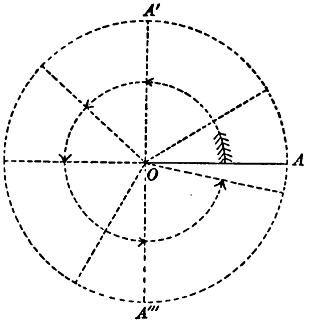
GONIOMETRY.

Goniometry is that branch of Trigonometry which treats of the trigonometric functions in general, and of their relations.

§ 1. ANGULAR MAGNITUDE.

Angles may be of any magnitude. — In practical applications of Trigonometry, the angles between zero and two right angles are the only ones it is necessary to consider. For theoretical purposes, however, an angle is conceived to be generated by the revolution of a line about a point, and hence *as having any magnitude whatever,* not only from 0° to 180°, but from 0° to 360°, and even to any number of degrees greater.

If the line *OA* be supposed to make one revolution about the point *O*, it will have described, at *A'*, an angular magnitude of 90°; at *A''*, an angular magnitude of 180°; at *A'''*, of 270°; and at *A* again, of 360°. If now it continue to revolve,

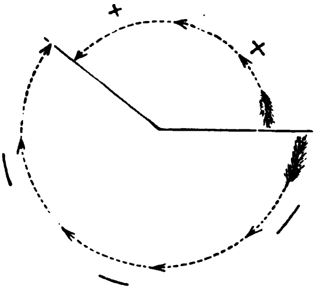


when it again arrives at A', it will have described an angle of $360° + 90° = 450°$, etc.

The fixed line is the *initial* line, and the revolving line is the *terminal* line. The normal motion of the revolving line is considered to be from right to left. Each of the four right angles, ACA', $A'CA''$, $A''CA'''$, and $A'''CA$, as comprising a quarter of the revolution, is called a *quadrant*. The quadrants are numbered *First*, *Second*, *Third*, and *Fourth*, in the order in which they are generated. If an angle is between 0° and 90°, it is said to be in the first quadrant; if the angle is greater than 90° and less than 180°, in the second quadrant; if greater than 180° and less than 270°, in the third; if greater than 270° and less than 360°, in the fourth, etc.

§ 2. SIGNS OF ANGLES.

An angle may be negative. — Angles conceived as generated from right to left are regarded as positive (the *plus* mark, however, not always prefixed); those conceived as generated by the opposite motion, from left to right, are regarded as negative, and are marked *minus*.

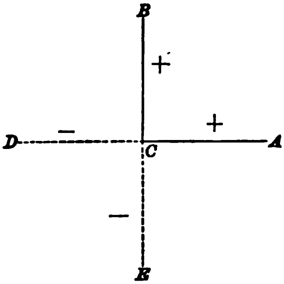


§ 3. SIGNS OF LINES.

A line may be negative. — As the *sign* of an angle denotes the *direction* in which the angle is generated, so the sign of a line, as distinguished from its value, denotes the direction in which the line is measured.

Distances in one direction from a given point are positive; those in the opposite direction from the point, negative.

Thus CA and CB are positive, CD and CE are negative.



Scholium 1.—The radius of a circle,—that is, the revolving line conceived as generating the angle,—since it is regarded as always extending in the same direction (from the centre toward the circumference), is regarded as always positive.

Scholium 2.—The symbols + and –, as thus used, do not, as in Algebra, indicate *operations*, but *qualities*. Prefixed to a number or letter denoting a line or angle, they have reference merely to direction. Thus, the line +4 is a line whose direction is positive, and whose length is 4 units; while the line –4 is a line whose direction is negative, and whose length, as before, is 4 units.

§ 4. EXERCISES.

1. Refer each of the following angles to its proper quadrant: 68°, 179°, 289°, 186°, 368°, 640° (= 360° + 280°), –30°, –80°, –188°.

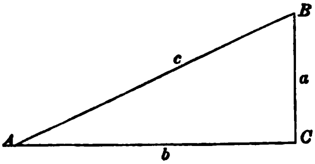
2. If $x = 60°$, and $y = 30°$, represent the following angles by diagrams: $x - 360°$, $x - 90°$, $x - 270°$, $x + y + 60°$.

§ 5. FUNDAMENTAL RELATIONS BETWEEN THE FUNCTIONS.

THEOREM I.

The square of the sine of an angle, plus the square of its cosine, is equal to 1.

For, by definition,



$$(1) \sin A = \frac{a}{c},$$

and $$(2) \cos A = \frac{b}{c};$$

whence, $$(3) \sin^2 A = \frac{a^2}{c^2},$$

and $$(4) \cos^2 A = \frac{b^2}{c^2};$$

$$(3) + (4) = (5) \sin^2 A + \cos^2 A = \frac{a^2 + b^2}{c^2} = 1,$$ since $a^2 + b^2 = c^2$.

Cor. I.—*The sine of an angle is equal to the square root of* 1 *minus the square of the cosine.*

For, from (5), $\sin^2 A = 1 - \cos^2 A$; whence

$$\sin A = \sqrt{1 - \cos^2 A}$$

Cor. II.—*The cosine of an angle is equal to the square root of* 1 *minus the square of the sine.*

THEOREM II.

The tangent of an angle is equal to its sine divided by its cosine.

For, (1) $\sin A = \frac{a}{c}$, and (2) $\cos A = \frac{b}{c}$.

(1) ÷ (2) = (3) $\frac{\sin A}{\cos A} = \frac{a}{b}$. But, by definition, $\tan A = \frac{a}{b}$;

$$\therefore \tan A = \frac{\sin A}{\cos A}.$$

Cor. — *The cotangent of an angle is equal to its cosine divided by its sine.*

For, by theorem,

$$\tan (90^\circ - A) = \frac{\sin (90^\circ - A)}{\cos (90^\circ - A)};$$

$$\therefore \cot A = \frac{\cos A}{\sin A}.$$

THEOREM III.

The tangent of an angle multiplied by its cotangent is equal to 1.

For, (1) $\tan A = \frac{a}{b}$, and (2) $\cot A = \frac{b}{a}$.

(1) × (2) = (3) $\tan A \cot A = \frac{a}{b} \times \frac{b}{a} = 1$.

Cor. I. — *The tangent of an angle is equal to the reciprocal of its cotangent.*

For, from (3),

$$\tan A = \frac{1}{\cot A}.$$

Cor. II. — *The cotangent of an angle is equal to the reciprocal of its tangent.*

THEOREM IV.

The secant of an angle is equal to the reciprocal of its cosine.

For, by definition,

$$\sec A = \frac{c}{b} = \frac{1}{\frac{b}{c}} = \frac{1}{\cos A}.$$

Cor. — *The cosecant of an angle is equal to the reciprocal of its sine.*

For,

$$\sec (90^\circ - A) = \frac{1}{\cos (90^\circ - A)}.$$

$$\therefore \csc A = \frac{1}{\sin A}.$$

THEOREM V.

The square of the secant of an angle is equal to 1 *plus the square of its tangent.*

For, from the diagram, we have

$$(1)\ c^2 = b^2 + a^2.$$

$(1) \div b^2 = (2)\ \frac{c^2}{b^2} = 1 + \frac{a^2}{b^2}$. But $\frac{c}{b} = \sec A$, and $\frac{a}{b} = \tan A$; $\therefore \sec^2 A = 1 + \tan^2 A.$

Cor. — *The square of the cosecant is equal to* 1 *plus the square of the cotangent.*

For,

$$\sec^2 (90° - A) = 1 + \tan^2 (90° - A).$$

$$\therefore \csc^2 A = 1 + \cot {}^2A.$$

THEOREM VI.

The versed sine of an angle is equal to 1 *minus its cosine.*

For, by definition,

$$\text{vers } A = \frac{c - b}{c} = 1 - \frac{b}{c} = 1 - \cos A.$$

Cor. — *The coversed sine of an angle is equal to* 1 *minus its sine.*

For,

$$\text{vers } (90° - A) = 1 - \cos (90° - A).$$

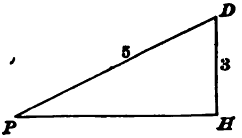
$$\therefore \text{cvs } A = 1 - \sin A.$$

§ 6. SUMMARY OF FUNDAMENTAL RELATIONS.

1. $\sin {}^2A + \cos {}^2A = 1.$
2. $\sin A = \sqrt{1 - \cos {}^2A}.$
3. $\cos A = \sqrt{1 - \sin {}^2A}.$
4. $\tan A = \dfrac{\sin A}{\cos A}$
5. $\cot A = \dfrac{\cos A}{\sin A}.$
6. $\tan A \cot A = 1.$
7. $\tan A = \dfrac{1}{\cot A}.$
8. $\cot A = \dfrac{1}{\tan A}.$
9. $\sec A = \dfrac{1}{\cos A}.$
10. $\csc A = \dfrac{1}{\sin A}.$
11. $\sec {}^2A = 1 + \tan {}^2A.$
12. $\csc {}^2A = 1 + \cot {}^2A.$
13. $\text{vers } A = 1 - \cos A.$
14. $\text{cvs } A = 1 - \sin A.$

§ 7. EXERCISES.

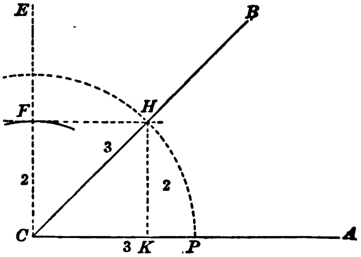
1. What functions are reciprocals of each other?



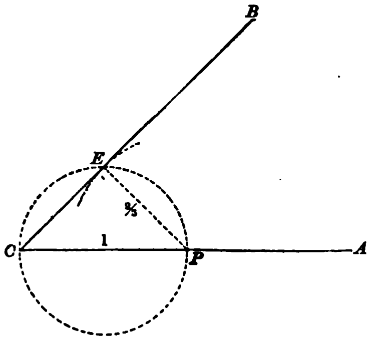
2. Given cvs $P = \frac{3}{5}$; find the remaining functions of P.

3. Given $\sin A = \frac{1}{3}$; find the remaining functions of A.
4. Given $\cos A = \frac{2}{3}$; find the remaining functions of A.
5. Given $\tan A = \frac{3}{2}$; find the remaining functions of A.
6. Given $\cot A - \frac{7}{8}$; find the remaining functions of A.
7. Given $\sec A = 3$; find the remaining functions of A.
8. Given $\csc A = \frac{7}{4}$; find the remaining functions of A.
9. Prove $(\sin A + \cos A)^2 + (\sin A - \cos A)^2 = 2$.
10. Prove $\sin {}^2A - \cos {}^2B = \sin {}^2B - \cos {}^2A$.
11. Prove $\tan A + \cot A = \sec A \csc A$.
12. Construct an angle whose sine $= \frac{2}{3}$.

First Method. — Draw an indefinite line AC, and let C be the angular point. Lay off $CP = 3$ units of measure, and from C, with a radius CP, describe an arc. At C erect a perpendicular, and take $CF = 2$ units of measure. Through F draw a line parallel to AC, intersecting the arc in H. Through H draw BC: then ACB is such an angle as is required; for $\sin ACB = \frac{HK}{CH} = \frac{2}{3}$.



Second Method.—Draw an indefinite line AC, and let C be the angular point. Lay off any portion CP assumed equal to *unity*, that is, the unit of measure; and on this as a diameter describe a circle. From centre P, with radius $= \frac{2}{3}\,CP$, describe an arc intersecting the circle in E. Through E draw BC: then ACB is such an angle as is required; for CEP is a right angle, and



$$\sin ACB = \frac{EP}{CP} = \frac{\frac{2}{3}}{1} = \tfrac{2}{3}.$$

13. Construct an angle whose sine $= \frac{1}{4}$.

14. Construct an angle whose cosine $= \frac{2}{3}$.

Suggestion.—Employing the same construction as in the second method, CPE is such an angle as is required; for,

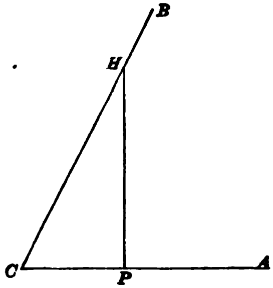
$$\cos CPE = \frac{\text{adjacent side}}{\text{hypothenuse}} = \frac{\frac{2}{3}}{1} = \tfrac{2}{3}.$$

Employing the construction of the first method, CHK is such an angle as is required. Or, if it is required to make C the angular point, erect a perpendicular at $\frac{2}{3}$ from C (second point of division), and through the point of its intersection with the arc draw a line from C.

15. Construct an angle whose cosine $= \frac{5}{13}$.

16. Construct an angle whose tangent $= 2$.

Solution. — Draw an indefinite line AC, and take CP = *unity*, assuming C as the angular point. At P, erect a perpendicular, and lay off a portion PH = *twice* CP. Through C and H draw BC: then ACB is such an angle as is required; for $\tan ACB = \frac{PH}{CP} = \frac{2}{1} = 2$.



17. Construct an angle whose tangent = $\frac{4}{5}$.

18. Construct an angle whose cotangent = 3.

19. Construct an angle whose secant = 2.

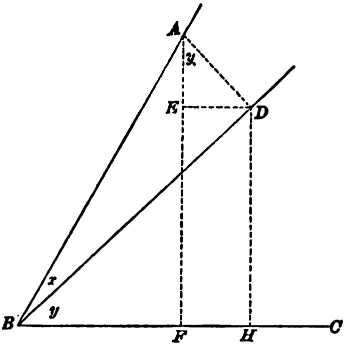
20. Prove $\sin^4 A + \cos^4 A = 1 - 2\sin^2 A \cos^2 A$.

21. Prove $\sin^3 A + \cos^3 A = (\sin A + \cos A)(1 - \sin A \cos A)$.

§ 8. FUNCTIONS OF THE SUM OR DIFFERENCE OF ANGLES.

THEOREM I.

The sine of the sum of two angles is equal to the product of the sine of the first by the cosine of the second, plus the product of the cosine of the first by the sine of the second.



Let ABD and DBC be the two angles denoted respectively by x and y: then will

$$\sin(x + y) = \sin ABC$$
$$= \sin x \cos y + \cos x \sin y.$$

For, from any point in AB, as A, draw AD perpendicular to BD. From the points A and D, let fall the perpendiculars AF and DH. Draw DE parallel to BC.

$$DAE = DBC, \text{ that is, } y.$$

$$\sin(x + y) = \sin ABC = \frac{AF}{AB} = \frac{AE + EF}{AB} = \frac{AE}{AB} + \frac{EF}{AB}$$
$$= \frac{AD}{AB} \times \frac{AE}{AD} + \frac{BD}{AB} \times \frac{DH}{BD}$$
$$= \sin x \cos y + \cos x \sin y.$$

Scholium. — It is noticed that the sum of the angles x and y is less than 90°. It is not thought best to cumber the demonstration with those cases (whose proof is entirely similar) in which the sum is greater than 90°. The result is of general application, provided we observe those values and signs of the functions shortly to be determined. The procedure is analogous to that constantly employed in Analytical Geometry, — the passage from the particular to the general.

THEOREM II.

The cosine of the sum of two angles is equal to the product of their cosines, minus the product of their sines.

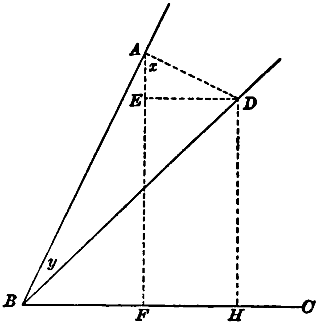
For, assuming the same construction as before,

$$\cos (x + y) = \cos ABC = \frac{BF}{AB} = \frac{BH - ED}{AB} = \frac{BH}{AB} - \frac{ED}{AB}$$
$$= \frac{BD}{AB} \times \frac{BH}{BD} - \frac{AD}{AB} \times \frac{ED}{AD}$$
$$= \cos x \cos y - \sin x \sin y.$$

THEOREM III.

The sine of the difference of two angles is equal to the product of the sine of the first by the cosine of the second, minus the product of the cosine of the first by the sine of the second.

Let $ABC = x$, $ABD = y$; then $DBC = x - y$. From any point D in DB, draw DA perpendicular to AB. From points A and D drop perpendiculars upon BC, and from D draw DE parallel to BC.



$$DAE = ABC, \text{ that is, } x.$$

$$\sin(x-y)=\sin DBC=\frac{DH}{BD}=\frac{AF-AE}{BD}=\frac{AF}{BD}-\frac{AE}{BD}$$
$$=\frac{AF}{AB}\times\frac{AB}{BD}-\frac{AE}{AD}\times\frac{AD}{BD}$$
$$=\sin x\cos y-\cos x\sin y.$$

THEOREM IV.

The cosine of the difference of two angles is equal to the product of their cosines, plus the product of their sines.

For, assuming the same construction,

$$\cos(x-y)=\cos DBC=\frac{BH}{BD}=\frac{BF+FH}{BD}=\frac{BF}{BD}+\frac{FH}{BD}$$
$$=\frac{BF}{AB}\times\frac{AB}{BD}+\frac{ED}{AD}\times\frac{AD}{BD}$$
$$=\cos x\cos y+\sin x\sin y.$$

THEOREM V.

The tangent of the sum of two angles is equal to the sum of their tangents, divided by 1 minus the product of their tangents.

For,

$$\tan(x+y)=\frac{\sin(x+y)}{\cos(x+y)}=\frac{\sin x\cos y+\cos x\sin y}{\cos x\cos y-\sin x\sin y}$$
$$=\frac{\dfrac{\sin x}{\cos x}+\dfrac{\sin y}{\cos y}}{1-\dfrac{\sin x\sin y}{\cos x\cos y}}=\frac{\dfrac{\sin x}{\cos x}+\dfrac{\sin y}{\cos y}}{1-\dfrac{\sin x}{\cos x}\times\dfrac{\sin y}{\cos y}}$$
$$=\frac{\tan x+\tan y}{1-\tan x\tan y};$$

dividing numerator and denominator by $\cos x \cos y$, and substituting $\tan x$ and $\tan y$ for $\frac{\sin x}{\cos x}$ and $\frac{\sin y}{\cos y}$.

THEOREM VI.

The cotangent of the sum of two angles is equal to the product of their cotangents minus 1, divided by the sum of their cotangents.

For,

$$\cot(x + y) = \frac{\cos(x + y)}{\sin(x + y)} = \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y + \cos x \sin y},$$

which, dividing both terms by $\sin x \sin y$, and reducing, $= \frac{\cot x \cot y - 1}{\cot x + \cot y}$.

THEOREM VII.

The tangent of the difference of two angles is equal to the difference of their tangents, divided by 1 plus the product of their tangents.

For,

$$\tan(x - y) = \frac{\sin(x - y)}{\cos(x - y)} = \frac{\sin x \cos y - \cos x \sin y}{\cos x \cos y + \sin x \sin y},$$

which, dividing both terms by $\cos x \cos y$, and reducing, $= \frac{\tan x - \tan y}{1 + \tan x \tan y}$.

THEOREM VIII.

The cotangent of the difference of two angles is equal to the product of their cotangents plus 1, divided by the cotangent of the second minus the cotangent of the first.

For,

$$\cot(x-y) = \frac{\cos(x-y)}{\sin(x-y)} = \frac{\cos x \cos y + \sin x \sin y}{\sin x \cos y - \cos x \sin y},$$

$$= \frac{\cot x \cot y + 1}{\cot y - \cot x}.$$

§ 9. SUMMARY OF RESULTS.

1. $\sin(x+y) = \sin x \cos y + \cos x \sin y.$
2. $\sin(x-y) = \sin x \cos y - \cos x \sin y.$
3. $\cos(x+y) = \cos x \cos y - \sin x \sin y.$
4. $\cos(x-y) = \cos x \cos y + \sin x \sin y.$
5. $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}.$
6. $\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}.$
7. $\cot(x+y) = \dfrac{\cot x \cot y - 1}{\cot x + \cot y}.$
8. $\cot(x-y) = \dfrac{\cot x \cot y + 1}{\cot y - \cot x}.$

§ 10. EXERCISES.

1. Given $\tan x = \frac{4}{3}$, and $\tan y = 7$; find $\tan(x+y)$.
2. Given $\tan x = \dfrac{1}{\sqrt{3}}$, and $\tan y = \dfrac{1}{\sqrt{15}}$; find $\sin(x+y)$.
3. Given $\sin x = \dfrac{1}{\sqrt{3}}$, and $\sin y = \dfrac{1}{\sqrt{2}} + \dfrac{1}{2\sqrt{3}}$; find $\cos(x+y)$.
4. Develop $\sin(45° + 30°)$.
5. Develop cos 105° by formula for the cosine of the sum.
6. Derive the formula for $\sin(x+y+z)$.

Suggestion. — Use formula of Th. I, substituting $y+z$ for y.

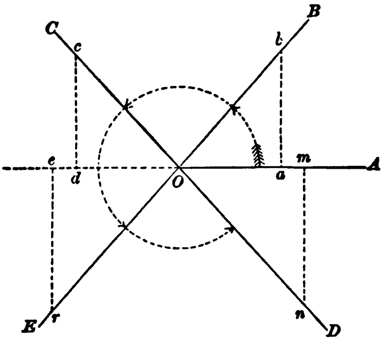
7. Derive the formula for $\tan(x+y+z)$.

§ 11. SIGNS OF THE FUNCTIONS.

1. *The trigonometric functions of an angle are ratios between the sides of a right triangle, in which, or with respect to which, the angle has a certain position.*

The right triangle is called the *triangle of reference.* It is formed by the sides of the angle, and a perpendicular drawn from any point in one of the sides to the other, or to the other produced.

2. From an inspection of the diagram, it appears that the triangle of reference for an *acute* angle is in the *first* quadrant of the revolution; for an angle greater than 90° but less than 180°, in the *second* quadrant; for an angle greater than 180° but less than 270°, in the *third* quadrant; for an angle greater than 270° but less than 360°, in the *fourth* quadrant. For an angle greater than 360°, the triangle of reference will evidently be the same as that of the excess.



3. We can now give the following general definitions of the trigonometric functions:

(1.) The **Sine** of any angle is the ratio of the *perpendicular* to the *hypothenuse* of the *triangle of reference.*

(2.) The **Cosine** of any angle is the ratio of the *base* to the *hypothenuse* of the *triangle of reference.*

(3.) The **Tangent** of any angle is the ratio of the *perpendicular* to the *base* of the *triangle of reference.*

(4.) The **Cotangent** of any angle is the ratio of the *base* to the *perpendicular* of the *triangle of reference.*

(5.) The **Secant** of any angle is the ratio of the *hypothenuse* to the *base* of the *triangle of reference.*

(6.) The **Cosecant** of any angle is the ratio of the *hypothenuse* to the *perpendicular* of the *triangle of reference.*

The definitions of the subsidiary functions, versed sine, and coversed sine, as given in § 6, are sufficiently general to apply to all angles.

4. In the first quadrant, all the sides of the reference triangle are positive. In the second quadrant, the base is negative by the convention of § 3; the hypothenuse being positive by § 3, Scholium 1. In the third quadrant, the base and the perpendicular are negative. In the fourth quadrant, the perpendicular is negative.

From these definitions result directly the following equations:

$$\sin AOB = \frac{ab}{Ob} = \frac{\text{positive quantity}}{\text{positive quantity}} = \text{positive quantity}.$$

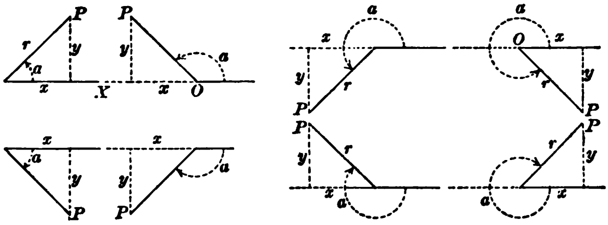
$$\sin AOC = \frac{cd}{Oe} = \frac{\text{positive quantity}}{\text{positive quantity}} = \text{positive quantity}.$$

$$\sin AOE = \frac{er}{Or} = \frac{\text{negative quantity}}{\text{positive quantity}} = \text{negative quantity}.$$

$$\tan AOE = \frac{er}{Oe} = \frac{\text{negative quantity}}{\text{negative quantity}} = \text{positive quantity}.$$

Etc., etc., etc.

Scholium. — The hypothenuse is also called the *distance* (r) of any point (P) on the terminal line; the base, or initial line, its *abscissa* (x); and the perpendicular, its *ordinate* (y).



Then:

sin = ordinate : distance = $y : r$.

cos = abscissa : distance = $x : r$.

tan = ordinate : abscissa = $y : x$.

cot = abscissa : ordinate = $x : y$.

sec = distance : abscissa = $r : x$.

csc = distance : ordinate = $r : y$.

§ 12. SUMMARY.

ANGLE.	SIN.	COS.	TAN.	COT.	SEC.	CSC.	VERS.	CVS.	
> 0 and $< 90°$	+	+	+	+	+	+	+	+	1st quad.
$> 90°$ and $< 180°$	+	−	−	−	−	+	+	+	2d quad.
$> 180°$ and $< 270°$	−	−	+	+	−	−	+	+	3d quad.
$> 270°$ and $< 360°$	−	+	−	−	+	−	+	+	4th quad.

Cor. I. — *The functions of any acute angle are positive.*

Cor. II. — *The sine and cosecant of obtuse angles are positive, while the secant, cosine, tangent, and cotangent are negative.*

Cor. III. — *The versed sine and coversed sine are always positive.*

Cor. IV. — *When one of two angles is greater than the other by four right angles, the functions of the two are the same.*

§ 13. FUNCTIONS OF NEGATIVE ANGLES.

In the formulas, $\sin(x - y) = \sin x \cos y - \cos x \sin y$, and $\cos(x - y) = \cos x \cos y + \sin x \sin y$, let $x = y$; then

$$\sin 0° = \sin x \cos x - \cos x \sin x = 0;$$

$$\text{and } \cos 0° = \cos^2 x + \sin^2 x = 1.$$

Putting $x = 0°$, we have, after substituting the functions of 0°,

$$\sin(-y) = -\sin y,$$

$$\cos(-y) = \cos y; \text{ whence}$$

$$\tan(-y) = \frac{\sin(-y)}{\cos(-y)} = \frac{-\sin y}{\cos y} = -\tan y;$$

$$\cot(-y) = \frac{\cos(-y)}{\sin(-y)} = \frac{\cos y}{-\sin y} = -\cot y;$$

$$\sec(-y) = \frac{1}{\cos(-y)} = \frac{1}{\cos y} = \sec y;$$

$$\csc(-y) = \frac{1}{\sin(-y)} = \frac{1}{-\sin y} = -\csc y;$$

$$\text{vers}(-y) = 1 - \cos(-y) = 1 - \cos y;$$

$$\text{cvs}(-y) = 1 - \sin(-y) = 1 + \sin y.$$

NOTE. — Let the student enunciate each of these results as a theorem.

§ 14. EXERCISES.

1. What signs belong to the different functions of 289°? of 368°? of 80°? of −188°?

2. In what quadrants must an angle be taken whose secant $= +\frac{5}{4}$?

3. In what quadrants must an angle be taken whose cosecant $= -2$?

4. In what quadrants must an angle be taken whose tangent $= -\frac{4}{5}$?

5. Given $\cos x = -\sqrt{\frac{1}{2}}$; find the remaining functions of x when x is taken in the first quadrant.

6. Given $\tan x = \sqrt{3}$; find the remaining functions of x when x is taken in the third quadrant.

§ 15. LIMITING VALUES OF THE FUNCTIONS.

We have found

$$\sin 0° = 0,$$
$$\cos 0° = 1.$$

Again,

$$\tan 0° = \frac{\sin 0°}{\cos 0°} = \frac{0}{1} = 0;$$
$$\cot 0° = \frac{1}{\tan 0°} = \frac{1}{0} = \infty;$$
$$\sec 0° = \frac{1}{\cos 0°} = \frac{1}{1} = 1;$$
$$\csc 0° = \frac{1}{\sin 0°} = \frac{1}{0} = \infty;$$
$$\text{vers } 0° = 1 - \cos 0° = 1 - 1 = 0;$$
$$\text{cvs } 0° = 1 - \sin 0° = 1 - 0 = 1.$$

Since the functions of an angle are equal to the co-functions of the complement,

$$\sin 90° = \cos (90° - 90°) = \cos 0° = 1;$$
$$\cos 90° = \sin (90° - 90°) = \sin 0° = 0;$$
$$\tan 90° = \frac{\sin 90°}{\cos 90°} = \frac{1}{0} = \infty;$$
$$\cot 90° = \frac{1}{\tan 90°} = \frac{1}{\infty} = 0.$$

. . . .

In the formulas, $\sin (x + y) = \sin x \cos y + \cos x \sin y$ and $\cos (x + y) = \cos x \cos y - \sin x \sin y$, let $x = y = 90°$; then

$$\sin 180° = 1 \times 0 + 0 \times 1 = 0;$$

$$\cos 180° = 0 \times 0 - 1 \times 1 = -1; \text{ whence}$$

$$\tan 180° = \frac{0}{-1} \qquad = -0;$$

$$\cot 180° = \frac{1}{0} \qquad = \infty;$$

$$\sec 180° = \frac{1}{-1} \qquad = -1.$$

. . . .

Let $x = 180°$, and $y = 90°$; then

$$\sin 270° = 0 \times 0 + (-1) \times 1 = -1;$$

$$\cos 270° = (-1) \times 0 - 0 \times 1 = 0.$$

. . . .

Let $x = y = 180°$; then

$$\sin 360° = 0 \times (-1) + (-1) \times 0 = 0;$$

$$\cos 360° = (-1) \times (-1) - 0 \times 0 = 1.$$

. . . .

To find the functions of an angle greater than four right angles, subtract from the angle 360°, or some multiple of 360°, and take the functions of the remaining angle.

§ 16. SUMMARY.

• ANGLE.	0°	90°	180°	270°	360°
Sine	+ 0	+ 1	+ 0	− 1	− 0
Cosine . . .	+ 1	+ 0	− 1	− 0	+ 1
Tangent . .	+ 0	+ ∞	− 0	− ∞	− 0
Cotangent. .	+ ∞	+ 0	− ∞	+ 0	− ∞
Secant . . .	+ 1	+ ∞	− 1	− ∞	+ 1
Cosecant . .	+ ∞	+ 1	+ ∞	− 1	− ∞
Versed Sine.	+ 0	+ 1	+ 2	+ 1	+ 0
Coversed Sine	+ 1	+ 0	+ 1	+ 2	+ 1

Cor. I.— *The sines of all angles are comprised within the limits +1 and −1; that is, the sine can never be greater than +1 nor less than −1.*

Cor. II. — *The cosines of all angles are comprised within the limits +1 and −1.*

Cor. III. — *The tangents of all angles are comprised within the limits* $+\infty$ *and* $-\infty$.

Cor. IV. — *The cotangents of all angles are comprised within the limits* $+\infty$ *and* $-\infty$.

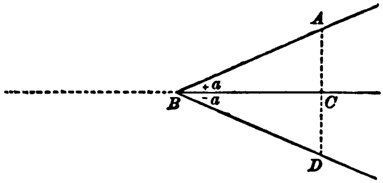
Cor. V. — *The secants and cosecants of all angles are comprised within the limits* $+1$ *and* $+\infty$, *and* -1 *and* $-\infty$.

Cor. VI. — *The limits of the versed sine and coversed sine are* 0 *and* 2.

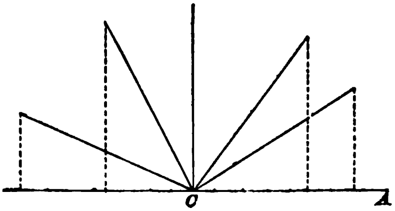
Scholium. — The limiting values have been determined for positive angles. In any given case, the functions of the negative angle may be determined from these by § 13.

§ 17. EXERCISES.

1. Trace the relations, geometrically, between the functions of positive angles and of equal negative angles, from the annexed diagram, in which the triangles of reference, *ABC* and *DBC*, are equal.



2. Determine, geometrically, the values of the functions at the limits 0°, 90°, 180°.



Suggestion. — Let AC revolve about the point C, and from its extremity, in each position, drop a perpendicular, forming a series of right triangles all having the same hypothenuse, AC. (What are the general definitions of the functions?)

3. Show:

(1.) That the *sine* and *cosine* of an *acute* angle are always less than 1;

(2.) That the *secant* and *cosecant* of an *acute* angle are always greater than 1;

(3.) That the *tangent* and *cotangent* of an *acute* angle may be < 1, $= 1$, or > 1;

(4.) That, as the acute angle increases from 0° to 90°, the *sine*, *versed sine*, *tangent*, and *secant increase*, while the *co*-functions *decrease;*

(5.) That the *sine* and *cosine* of an *obtuse* angle (that is, one $> 90°$ and $< 180°$) are always less than 1 numerically.

(6.) That the *tangent* and *cotangent* of an *obtuse* angle may be $= -1$, > -1, or < -1.

(7.) That the *secant* and *cosecant* of an *obtuse* angle are always greater than 1 numerically, the secant being > -1, the cosecant $> +1$.

(8.) That, as the obtuse angle approaches 180°, its *sine* and *tangent decrease*, the former both algebraically and numerically, the latter numerically; while the *cosine* and *cotangent increase* numerically.

§ 18. FUNCTIONS OF $(n\ 90° \pm y)$.

In the formulas,

$$(1) \qquad \sin(x-y) = \sin x \cos y - \cos x \sin y,$$

$$(2) \qquad \cos(x-y) = \cos x \cos y + \sin x \sin y,$$

let $x = 90°$; then

$$(a.) \begin{cases} \sin(90° - y) = \cos y, \\ \cos(90° - y) = \sin y, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{cases}$$

In the formulas,

$$(3) \quad \sin (x + y) = \sin x \cos y + \cos x \sin y,$$

$$(4) \quad \cos (x + y) = \cos x \cos y - \sin x \sin y,$$

let $x = 90°$; then

$$(b.) \left\{ \begin{array}{lrl} \sin (90° + y) & = & \cos y, \\ \cos (90° + y) & = - & \sin y, \text{ whence} \\ \tan (90° + y) & = - & \cot y, \\ \cot (90° + y) & = - & \tan y, \\ \sec (90° + y) & = - & \csc y, \\ \csc (90° + y) & = & \sec y, \\ . \quad . \quad . & . & . \quad . \end{array} \right.$$

In (1) and (2), let $x = 180°$; then

$$(c.) \left\{ \begin{array}{lrl} \sin (180° - y) & = & \sin y, \\ \cos (180° - y) & = - & \cos y, \text{ whence} \\ \tan (180° - y) & = - & \tan y, \\ \cot (180° - y) & = - & \cot y, \\ \sec (180° - y) & = - & \sec y, \\ \csc (180° - y) & = - & \csc y, \\ . \quad . \quad . & . & . \quad . \end{array} \right.$$

In (3) and (4), let $x = 180°$; then

$$d.) \left\{ \begin{array}{lrl} \sin (180° + y) & = - & \sin y, \\ \cos (180° + y) & = - & \cos y, \text{ whence} \\ \tan (180° + y) & = & \tan y, \\ \cot (180° + y) & = & \cot y, \\ \sec (180° + y) & = - & \sec y, \\ \csc (180° + y) & = - & \csc y, \\ . \quad . \quad . & . & . \quad . \quad . \end{array} \right.$$

In (1) and (2), let $x = 270°$; then

$$(e.)\quad \left\{\begin{array}{lll}
\sin\,(270° - y) = -\cos y, & \\
\cos\,(270° - y) = -\sin y, & \text{whence} \\
\tan\,(270° - y) = \cot y, & \\
\cot\,(270° - y) = \tan y, & \\
\sec\,(270° - y) = -\csc y, & \\
\csc\,(270° - y) = -\sec y, & \\
\ldots\ldots &
\end{array}\right.$$

In (3) and (4), let $x = 270°$; then

$$(f.)\quad \left\{\begin{array}{lll}
\sin\,(270° + y) = -\cos y, & \\
\cos\,(270° + y) = \sin y, & \text{whence} \\
\tan\,(270° + y) = -\cot y, & \\
\cot\,(270° + y) = -\tan y, & \\
\sec\,(270° + y) = \csc y, & \\
\csc\,(270° + y) = -\sec y, & \\
\ldots\ldots &
\end{array}\right.$$

In (1) and (2), let $x = 360°$; then

$$(g.)\quad \left\{\begin{array}{lll}
\sin\,(360° - y) = -\sin y, & \\
\cos\,(360° - y) = \cos y, & \text{whence} \\
\tan\,(360° - y) = -\tan y, & \\
\cot\,(360° - y) = -\cot y, & \\
\sec\,(360° - y) = \sec y, & \\
\csc\,(360° - y) = -\csc y, & \\
\ldots\ldots &
\end{array}\right.$$

In (3) and (4), let $x = 330°$; then

$$(h.)\quad \left\{\begin{array}{lll}
\sin\,(360° + y) = \sin y, & \\
\cos\,(360° + y) = \cos y, & \text{whence} \\
\tan\,(360° + y) = \tan y, & \\
\ldots\ldots &
\end{array}\right.$$

Cor. I. — *The sine and cosecant of an angle are equal to the cosine and secant of the excess of the angle above 90°; and the cosine, tangent, cotangent, and secant of an angle are equal to the negatives of the sine, cotangent, tangent, and cosecant of the excess.* See formulas (*b*).

Cor. II. — *The sine and cosecant of the supplement of an angle are the same as those of the angle itself; and the cosine, tangent, cotangent, and secant of the supplement are the negatives of those angles.* See formulas (*c*).

Cor. III. — *The functions of* $360^\circ - y$ *are the same as those of* $-y$.

Cor. IV. — *The functions of* $360^\circ + y$ *are the same as those of* y.

Scholium 1. — Functions of angles greater than 90° can thus be found in terms of functions of angles less than 90°.

Thus,

$$\left.\begin{array}{l} \sin 120^\circ = \sin\ (90^\circ + 30^\circ) = \cos\ 30^\circ; \\ \cos 290^\circ = \cos\ (270^\circ + 20^\circ) = \sin\ 20^\circ; \\ \tan 165^\circ = \tan\ (180^\circ - 15^\circ) = \tan -15^\circ; \\ \sec 150^\circ = \sec\ (180^\circ - 30^\circ) = \sec\ 30^\circ. \end{array}\right\} \quad (?)$$

Scholium 2. — By means of formulas (*d*), y being acute and given, we may obtain the functions of angles between 180° and 270°.

§ 19. EXERCISES.

1. How many angles, each less than 180°, correspond to a given sine?

2. How many angles, less than 180°, correspond to a given tangent?

3. What angles have their tangents each 1? each -1?

4. Construct in the third quadrant an angle whose sine $= -\frac{1}{2}$. Does any other angle less than 360° correspond to the same sine?

5. What are the sines of the several functions of $-1722°$, $-2000°$, 1178°?

6. Find a general expression for all the angles which have a given sine.

Solution. — The *positive* angles are x and $180° - x$ (Cor. II.), and those angles formed by adding any multiple of 360° to x or to $180° - x$ (Cor. IV.); that is, angles included in the expressions $n\ 360° + x$ and $n\ 360° + (180° - x)$, where n is zero or any *positive* integer. The *negative* angles are, $-(180° + x)$ [§ 13, and § 18 (*d*)], and $-(360° - x)$ [§ 13 and § 18, Cor. III.]; that is, the angles included in the expressions $n\ 360° - (180° + x)$ and $n\ 360° - (360° - x)$, where n is zero or any *negative* integer.

7. Find a general expression for all the angles which have a given cosine.

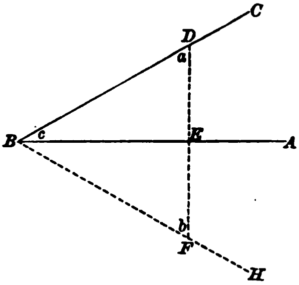
Ans. — *Positive* $= n\ 360° + x$, $n\ 360° + (360° - x)$; *Negative* $= n\ 360° - x$, $n\ 360° - (360° - x)$; where n in the first instance is integral and positive, in the second integral and negative.

8. Find a general expression for all the angles which have a given tangent.

§ 20. VALUES OF FUNCTIONS OF PARTICULAR ANGLES.

PROBLEM I.

To find the numerical values of the functions of 30°.



Let $ABC = 30°$. From any point in BC, as D, let fall a perpendicular upon AB, forming the reference triangle EBD. Produce DE to F, making $EF = DE$; and through F draw BH: then triangle EBF = triangle EBD.

Angle $DBF = 60°$;

$\therefore\ a = b = 60°$; triangle DEF is equilateral;

$\therefore\quad DB = DF = 2\,DE.$

Now,

$$(1)\ \sin 30° = \sin e = \frac{DE}{BC} = \frac{DE}{2\,DE} = \tfrac{1}{2}.$$

$$(2)\ \cos 30° = \sqrt{1 - \sin^2 30°} = \sqrt{1 - \tfrac{1}{4}} = \sqrt{\tfrac{3}{4}} = \tfrac{1}{2}\sqrt{3}.$$

$$(3)\ \tan 30° = \frac{\sin 30°}{\cos 30°} = \frac{\tfrac{1}{2}}{\tfrac{1}{2}\sqrt{3}} = \frac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3}.$$

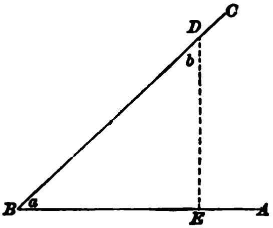
§ 5, *Th. II.*

$$(4)\ \cot 30° = \frac{\sqrt{3}}{1} = \sqrt{3},$$

§ 5, *Th. III, Cor. II.*

PROBLEM II.

To find the numerical values of the functions of 45°.



Let $ABC = 45°$. From any point in BC, as D, draw DE perpendicular to AB.

$$\text{Since } a = 45°,\ b = 45°;$$

$$\therefore BE = DE.$$

$$\overline{BD}^2 = \overline{BE}^2 + \overline{DE}^2$$

$$= 2\,\overline{DE}^2 = 2\,\overline{BE}^2;$$

$$\therefore BE = \frac{BD}{\sqrt{2}} = DE.$$

Now,

$$(1)\ \sin 45° = \sin a = \frac{DE}{BD} = \frac{\frac{BD}{\sqrt{2}}}{BD} = \frac{1}{\sqrt[2]{2}} = \tfrac{1}{2}\sqrt{2}.$$

$$(2)\ \cos 45° = \sqrt{1 - \sin^2 45°} = \sqrt{1 - \tfrac{1}{2}} = \sqrt{\tfrac{1}{2}} = \tfrac{1}{2}\sqrt{2}.$$

$$(1) \div (2) = (3)\ \tan 45° = \frac{\frac{1}{\sqrt{2}}}{\frac{1}{2}\sqrt{2}} = \frac{2}{2} = 1;$$

or, geometrically, $\tan 45° = \tan a = \dfrac{DE}{BE} = \dfrac{DE}{DE} = 1.$

$$(4)\ \cot 45° = \frac{1}{\tan 45°} = \frac{1}{1} = 1.$$

. . . .

PROBLEM III.

To find the numerical values of the functions of 60°.

(1) $\sin 60° = \cos (90° - 60°) = \cos 30° = \frac{1}{2}\sqrt{3}.$

(2) $\cos 60° = \sin (90° - 60°) = \sin 30° = \frac{1}{2}.$

(3) $\tan 60° = \cot (90° - 60°) = \cot 30° = \sqrt{3}.$

. . . .

PROBLEM IV.

To find the numerical values of the functions of 120°, 135°, and 150°.

(1) $\sin 120° = \sin (180° - 60°) = \sin 60°$

$= \frac{1}{2}\sqrt{3}$ . . § 18, (*c*), *Cor. II*, *Sch. 1.*

(2) $\sin 135° = \sin (180° - 45°) = \sin 45° = \frac{1}{2}\sqrt{2}.$

(3) $\sin 150° = \sin (180° - 30°) = \sin 30° = \frac{1}{2}.$

(4) $\cos 120° = \cos (180° - 60°)$

$= -\cos 60° = -\frac{1}{2}$ . . § 18, (*c*), *Cor. II.*

. . . .

PROBLEM V.

To find the numerical values of the functions of 210°, 225°, and 240°.

(1) $\sin 210° = \sin (180° + 30°) = -\sin 30°$

$= \frac{1}{2}$ . . . . § 18, *Sch. II.*

(2) $\cos 240° = \cos (180° + 60°) = -\cos 60° = -\frac{1}{2}.$

. . . .

PROBLEM VI.

To find the numerical values of the functions of 300°, 315°, and 330°.

(1) $\sin 300° = \sin(270° + 30°) = -\cos 30°$

$$= -\frac{\sqrt{3}}{2} \quad \ldots\ldots\ldots \quad \S\ 18,\ (f).$$

(2) $\sin 315° = \sin(270° + 45°) = -\cos 45° = -\sqrt{2}.$

(3) $\sin 330° = \sin(270° + 60°) = -\cos 60° = -\frac{1}{2}.$

. . . .

§ 21. SUMMARY.

ANGLE.	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	-0
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	-0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	-0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	-0
Cotangent	∞	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	$-\frac{\sqrt{3}}{3}$	-1	$-\sqrt{3}$	$-\infty$	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	$-\frac{\sqrt{3}}{3}$	-1	$-\sqrt{3}$	$-\infty$
Secant	1	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$	2	∞	-2	$-\sqrt{2}$	$-\frac{2\sqrt{3}}{3}$	-1	$-\frac{2\sqrt{3}}{3}$	$-\sqrt{2}$	-2	$-\infty$	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
Cosecant	∞	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	∞	-2	$-\sqrt{2}$	$-\frac{2\sqrt{3}}{3}$	-1	$-\frac{2\sqrt{3}}{3}$	$-\sqrt{2}$	-2	$-\infty$

§ 22. EXERCISES.

1. Find the values of $\sin 195°$, $\cot 420°$, $\tan 345°$, $\sin(-240°)$, $\cos(-300°)$, $\tan(-120°)$.

2. Solve the following equations with reference to $x < 90°$:

(1.) $3 \sin x = 2 \cos^2 x$.

Solution: $\cos^2 x = 1 - \sin^2 x$ . . . . § 5, *Th. I.*

$\therefore 3 \sin x = 2(1 - \sin^2 x) = 2 - 2 \sin^2 x$.

$\therefore 2 \sin^2 x + 3 \sin x = 2$.

$\sin^2 x + \frac{3}{2} \sin x = 1$.

$\therefore \sin x = -\frac{3}{4} \pm \sqrt{1 + \frac{9}{16}} = -\frac{3}{4} \pm \frac{5}{4} = \frac{2}{4} = \frac{1}{2}$,

the second root rejected (why?).

$\therefore x = 30°$.

(2.) $3 \csc^2 x + 8 \sin^2 x = 10$.

(3.) $\sec x \tan x = 2\sqrt{3}$.

(4.) $6 \cot^2 x - 4 \cos^2 x = 1$.

(5.) $\sin x + \cos x = \frac{1}{3}$.

(6.) $\sin x + \cos x = \dfrac{1}{\sqrt{2}}$.

3. Solve the following with reference to $x < 90°$ and $y < 90°$:

(1.) $$\frac{\sin x}{\sin y} = \sqrt{2},\ \frac{\cos x}{\cos y} = \frac{\sqrt{2}}{\sqrt{3}}.$$

Solution: $$\frac{\sin^2 x}{\sin^2 y} = 2, \text{ and } \frac{\cos^2 x}{\cos^2 y} = \frac{2}{3}.$$

$\therefore \sin^2 x = 2 \sin^2 y$, and $\cos^2 x = \frac{2}{3} \cos^2 y$.

$\therefore\ 1 - \sin^2 x = \frac{2}{3}(1 - \cos^2 y) = \frac{2}{3} - \frac{2}{3} \cos^2 y$.

But $1 - \sin^2 x = 1 - 2 \sin^2 y$.

$\therefore 1 - 2 \sin^2 y = \frac{2}{3} - \frac{2}{3} \cos^2 y$.

$\therefore 1 - \frac{4}{3} \sin^2 y = \frac{2}{3}$, $\quad \therefore \frac{4}{3} \sin^2 y = \frac{1}{3}$.

$\therefore \sin^2 y = \frac{1}{4}$.

$\therefore \sin y = \frac{1}{2}$, $\quad \therefore y = 30°$.

But $\sin^2 x = 2 \sin^2 y = 2 \times \frac{1}{4} = \frac{1}{2}$.

$\therefore \sin x = \sqrt{\frac{1}{2}} = \frac{1}{2}\sqrt{2}$, $\quad \therefore x = 45°$.

(2.) $\dfrac{\sin x}{\cos y} = \dfrac{\sqrt{3}}{\sqrt{2}}, \quad \dfrac{\cos x}{\sin y} = \dfrac{1}{\sqrt{2}}.$

(3.) $\sin x + \sin y = \sqrt{2}, \quad \sin^2 x + \sin^2 y = 1.$

4. If cosecant x = secant y, what is the relation between x and y?

5. What angles less than 360° have a cosine $= -\frac{1}{2}$?

6. What angles less than 360° will satisfy the equations $\sin^2 x = \frac{1}{4}$, $\cos^2 x = \frac{1}{2}$, $\tan^2 x = 3$?

7. Find a general expression for the angles which will satisfy $\sin x = \frac{1}{2}$.

§ 23. FUNCTIONS OF MULTIPLES OF ANGLES.

In the formulas,

$$\sin(x + y) = \sin x \cos y + \cos x \sin y,$$

and $$\cos(x + y) = \cos x \cos y - \sin x \sin y,$$

let $$y = x;$$

then (1) $$\sin 2x = \sin x \cos x + \cos x \sin x,$$

$$= (\sin x + \sin x) \cos x$$

and $$= 2 \sin x \cos x,$$

(2) $$\cos 2x = \cos^2 x - \sin^2 x.$$

In the formulas,

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y},$$

and $$\cot(x + y) = \frac{\cot x \cot y - 1}{\cot x + \cot y},$$

let $$y = x;$$

then (3) $$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x},$$

and (4) $$\cot 2x = \frac{\cot^2 x - 1}{2 \cot x}.$$

Letting $y = 2x$, we get

$$(5) \quad \sin 3x = 2 \sin 2x \cos x - \sin x,$$

$$(6) \quad \cos 3x = 2 \cos 2x \cos x - \cos x,$$

$$(7) \quad \tan 3x = \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}.$$

. . . .

Hence, having found the function of an angle, as x, we may, by successive substitution, easily find the corresponding function of *twice* the angle, *three* times the angle, etc.

Substituting in (1), (2), (5), and (6),

$$\cos x = \sqrt{1 - \sin^2 x},$$

we get

$$\sin 2x = 2 \sin x \sqrt{1 - \sin^2 x},$$

$$\sin 3x = 3 \sin x - 4 \sin^3 x,$$

$$\cos 2x = 1 - 2 \sin^2 x,$$

$$\cos 3x = (1 - 4 \sin^2 x) \sqrt{1 - \sin^2 x}.$$

Substituting (3) in (7),

$$\tan 3x = \frac{3 \tan x - \tan^3 x}{1 - 6 \tan^2 x + \tan^4 x}$$

. . . .

§ 24. EXERCISES.

1. Find the sine and tangent of 18°.

Suggestion. — Let $x = 18°$, then $2x = 36°$, $3x = 54°$, and $5x = 90°$; $\therefore \sin 2x = \cos 3x$. Develop both members, and solve the resulting equation, $4 \sin^2 x + 2 \sin x = 1$.

2. From the sine and tangent of 18° find the sine and tangent of 36° and 72°.

Ans. $\tan 72° = \dfrac{\sqrt{10 + 2\sqrt{5}}}{\sqrt{5} - 1}$.

3. Given $\tan x = 1 + \sqrt{2}$, to find $\cos^2 x$, $\sin^2 x$, and $\cos 2x$.
4. Given $\tan 2x = 3 \tan x$, to find x.
5. If $\tan x + \sec x = \frac{3}{2}$, prove that $\sin x = \frac{5}{13}$.
6. Find x and y from $\tan x \tan y = 1$, $\tan^2 x + \tan^2 y = \frac{10}{3}$.

§ 25. FUNCTIONS OF HALF-ANGLES.

$$(1) \quad \cos^2 x + \sin^2 x = 1 \quad . \quad . \quad . \quad . \quad . \quad \S\ 6.$$

$$(2) \quad \cos^2 x - \sin^2 x = \cos 2x \quad . \quad \S\ 23,\ (2).$$

$$(1) + (2) = (3) \qquad 2 \cos^2 x = 1 + \cos 2x.$$

$$(1) - (2) = (4) \qquad 2 \sin^2 x = 1 - \cos 2x.$$

Observe that x, in the first member of (3) and (4), represents an angle equal to *one half* the angle in the second member. If $2x$ represents any angle a, x represents *one half* the same angle. Hence,

$$(5) \qquad 2 \sin^2 \tfrac{1}{2} a = 1 - \cos a,$$

$$\text{and } (6) \qquad 2 \cos^2 \tfrac{1}{2} a = 1 + \cos a;$$

$$\therefore \quad (7) \qquad \sin \tfrac{1}{2} a = \sqrt{\frac{1 - \cos a}{2}},$$

$$\text{and } (8) \qquad \cos \tfrac{1}{2} a = \sqrt{\frac{1 + \cos a}{2}};$$

$$\therefore \quad (9) \qquad \tan \tfrac{1}{2} a = \sqrt{\frac{1 - \cos a}{1 + \cos a}}.$$

. . . .

Scholium 1.—By these formulas, the cosine of an angle being given, we may calculate the sine, cosine, and tangent of its half, then of its fourth, of its eighth, etc., by successive substitution of the values.

[Let the student enunciate (7), (8), and (9) as theorems. Thus: *The sine of half an angle is equal to the square root of half the difference between 1 and the cosine of the angle.* Or thus: *The sine of an angle is equal to the square root of half the difference between 1 and the cosine of twice the angle.*]

Scholium 2. — (7) and (8) may be employed to solve an oblique triangle when the three sides are given.

§ 26. EXERCISES.

1. From the functions of 30° find those of 15° and 7° 30′.
2. From the functions of 15° find those of 75°.
3. From the functions of 90° find those of 45°.
4. From the functions of 45° find those of 22° 30′.
5. From the functions of $22\frac{1}{2}°$ find those of $67\frac{1}{2}°$.
6. From $\sin x = \frac{2}{5}$ find $\sin \frac{1}{2} x$ and $\cos \frac{1}{2} x$.
7. Find the cosine of 11° 15′.
8. Find the sine and tangent of 9°.
9. Find the sine and cosine of 3°.

   Suggestion: $\text{Sin } 3° = \sin (18° - 15°)$.

10. Find the sine of 6° and 24°.
11. Find the sine of 33°.

    Suggestion: $\text{Sin } 33° = \sin (18° + 15°)$.

§ 27. CONSEQUENCES OF THEOREMS I, II, III, AND IV, § 8.

Adding the formulas,

$$(c) \quad \sin (x + y) = \sin x \cos y + \cos x \sin y,$$

$$(d) \quad \sin (x - y) = \sin x \cos y - \cos x \sin y,$$

we get

(1) $\sin(x+y) + \sin(x-y) = 2\sin x\cos y.$

Subtracting,

(2) $\sin(x+y) - \sin(x-y) = 2\cos x\sin y.$

Adding the formulas,

(e) $\cos(x+y) = \cos x\cos y - \sin x\sin y,$

(f) $\cos(x-y) = \cos x\cos y + \sin x\sin y,$

we get

(3) $\cos(x+y) + \cos(x-y) = 2\cos x\cos y.$

Subtracting,

(4) $\cos(x+y) - \cos(x-y) = -2\sin x\sin y.$

[Let the student render into ordinary language formulas (1), (2), (3), and (4). Thus: *The sine of the sum of two angles, plus the sine of their difference, is equal to the product of twice the sine of the first by the cosine of the second.*]

If we put

$$x + y = a,$$

and $x - y = b,$

then $2x = a + b, \quad \therefore x = \frac{1}{2}(a+b);$

and $2y = a - b, \quad \therefore y = \frac{1}{2}(a-b).$

Substituting in (1), (2), (3), and (4), we have

(5) $\sin a + \sin b = 2\sin\frac{1}{2}(a+b)\cos\frac{1}{2}(a-b).$

(6) $\sin a - \sin b = 2\cos\frac{1}{2}(a+b)\sin\frac{1}{2}(a-b).$

(7) $\cos a + \cos b = 2\cos\frac{1}{2}(a+b)\cos\frac{1}{2}(a-b).$

(8) $\cos a - \cos b = -2\sin\frac{1}{2}(a+b)\sin\frac{1}{2}(a-b).$

[Let these equations be enunciated as theorems. Thus: *The sum of the sines of any two angles is equal to the product of twice the sine of half the sum of the angles by the cosine of half their difference.* These formulas are very useful, especially in logarithmic computation, in transforming a sum or difference into a product.]

$$(5) \div (6) = \quad (9) \quad \frac{\sin a + \sin b}{\sin a - \sin b} = \tan \tfrac{1}{2}(a+b) \cot \tfrac{1}{2}(a-b)$$

$$= \frac{\tan \frac{1}{2}(a+b)}{\tan \frac{1}{2}(a-b)}.$$

$$(5) \div (7) = (10) \quad \frac{\sin a + \sin b}{\cos a + \cos b} = \tan \tfrac{1}{2}(a+b).$$

$$(5) \div (8) = (11) \quad \frac{\sin a + \sin b}{\cos a - \cos b} = -\cot \tfrac{1}{2}(a-b).$$

$$(6) \div (7) = (12) \quad \frac{\sin a - \sin b}{\cos a + \cos b} = \tan \tfrac{1}{2}(a-b).$$

$$(6) \div (8) = (13) \quad \frac{\sin a - \sin b}{\cos a - \cos b} = -\cot \tfrac{1}{2}(a+b).$$

By formula (1), § 23, *the sine of an angle is equal to the product of twice the sine of half the angle by the cosine of half the angle.* Hence,

$$(14) \quad \sin(a+b) = 2 \sin \tfrac{1}{2}(a+b) \cos \tfrac{1}{2}(a+b),$$

and

$$(15) \quad \sin(a-b) = 2 \sin \tfrac{1}{2}(a-b) \cos \tfrac{1}{2}(a-b).$$

$$(5) \div (14) = (16) \quad \frac{\sin a + \sin b}{\sin(a+b)} = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)}.$$

$$(5) \div (15) = (17) \quad \frac{\sin a + \sin b}{\sin(a-b)} = \frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}(a-b)}.$$

$$(6) \div (14) = (18) \quad \frac{\sin a - \sin b}{\sin (a + b)} = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)}.$$

$$(6) \div (15) = (19) \quad \frac{\sin a - \sin b}{\sin (a - b)} = \frac{\cos \frac{1}{2}(a + b)}{\cos \frac{1}{2}(a - b)}.$$

$$(7) \div (14) = (20) \quad \frac{\cos a + \cos b}{\sin (a + b)} = \frac{\cos \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)}.$$

$$(7) \div (15) = (21) \quad \frac{\cos a + \cos b}{\sin (a - b)} = \frac{\cos \frac{1}{2}(a + b)}{\sin \frac{1}{2}(a - b)}.$$

$$(8) \div (14) = (22) \quad \frac{\cos a - \cos b}{\sin (a + b)} = \frac{\sin \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)}.$$

$$(8) \div (15) = (23) \quad \frac{\cos a - \cos b}{\sin (a - b)} = \frac{\sin \frac{1}{2}(a + b)}{\cos \frac{1}{2}(a - b)}.$$

Dividing formulas (*c*), (*d*), (*e*), and (*f*) by $\cos x \cos y$, and recollecting that $\tan x = \frac{\sin x}{\cos x}$, we get

$$(24) \quad \frac{\sin (x + y)}{\cos x \cos y} = \tan x + \tan y;$$

$$(25) \quad \frac{\sin (x - y)}{\cos x \cos y} = \tan x - \tan y;$$

$$(26) \quad \frac{\cos (x + y)}{\cos x \cos y} = 1 - \tan x \tan y;$$

$$(27) \quad \frac{\cos (x - y)}{\cos x \cos y} = 1 + \tan x \tan y.$$

NOTE. — Let (24), (25), (26), and (27) be expressed in general terms.

§ 28. EXERCISES.

1. Find x from the equation $\sin 7x - \sin x = \sin 3x$.

Suggestion: $$\text{Sin } 7x - \sin x = \sin (4x + 3x) - \sin (4x - 3x)$$
$$= 2 \cos 4x \sin 3x,$$

since *the sine of the sum of two angles, minus the sine of their difference, is equal to the product of twice the cosine of the first by the sine of the second.*

2. Find x from $\tan x = \tan 4x$.

Suggestion: By transposition and change of signs,

$$\tan 4x - \tan x = 0.$$

But $$\tan 4x - \tan x = \frac{\sin (4x - x)}{\cos 4x \cos x} = \frac{\sin 3x}{\cos 4x \cos x}$$
$$= \frac{3 \sin x - 4 \sin^3 x}{\cos 4x \cos x} = 0 \quad . \ . \ (25), \text{ and § } 23.$$

3. Given $\sin (x + y) = \cos (x - y)$, to find x in terms of y.

4. Prove $\cos (60° + x) + \cos (60° - x) = \cos x$.

5. Prove $\cos 22° 30' - \cos 67° 30' = \sqrt{\frac{2 - \sqrt{2}}{2}}$.

[See formula (8).]

6. Adapt to logarithmic computation $\frac{\sin 68° - \sin 35°}{\sin (68° - 35°)}$, $\cos 38° - \cos 54°$, and $\frac{\sin 49° 18' + \sin 34° 48'}{\cos 34° 48' - \cos 49° 18'}$.

§ 29. INVERSE FUNCTIONS.

When two quantities are so related that a change in one causes a change in the other, the second, as we have seen, is a *function* of the first. Thus, $\sqrt{x}$ is a function of x; for if x changes, the *square root* of x changes: the circumference of a circle is a function of the radius; for if the radius varies, the circumference varies: the sine of an angle is a function of the angle; for if the angle changes, the sine changes. It is in this sense that the *ratios* of the *sides* of an angle are called the *trigonometric functions* of the angle.

If we consider the angle and any one of its functions as two *variables*, the angle will be the *independent* variable, and the function the *dependent* variable; since to the former any arbitrary values may be assigned, and, from the connection between the two, the corresponding values of the latter deduced.

Thus, in the equation $y = \sin x$, if x (the angle) changes, y (the function) changes; but the change of y *depends* upon the antecedent change of x.

It is important to remember that, in the form $y = \sin x$, y denotes the *function* (or *ratio*), and x denotes the *angle;* and that, of these two, the function is the thing mainly thought of, the x merely indicating *what* function, as the sine of 30°, of 75°, etc. Similarly, in the expressions $\tan x$ and $\sec x$ (which are functions of x), the principal object of thought is evidently the function.

If, now, we wish to invert the order of thought, — that is, if we wish to regard the angle as the *dependent* and the function as the *independent* variable, — the following notation is adopted:

$$x = \sin^{-1} y,\ \cos^{-1} z,\ \tan^{-1} n,\ \sec^{-1} \frac{a}{b};$$

which are read, x equals an angle whose sine is y, or x *is* an angle whose sine is y; x equals an angle whose cosine is z; x equals an angle whose tangent is n; y equals an angle whose secant is $\frac{a}{b}$. In this notation, the thing mainly thought of is the angle.

The expressions $\sin^{-1} y$, $\sec^{-1} \frac{a}{b}$, $\tan^{-1}(-1)$, $\tan^{-1} \frac{3}{4}$, etc., are called the *inverse* functions, or *anti*-functions. They may stand alone, and are read: an angle whose sine is y; an angle whose secant is $\frac{a}{b}$; an angle whose tangent is -1; an angle whose tangent is $\frac{3}{4}$, etc.[1]

Any relation shown to exist among the trigonometric functions can be expressed inversely. Thus,

$$\tan x = \frac{\sin x}{\cos x} = \frac{\sin}{\cos} x = m$$

may be expressed

$$x = \tan^{-1} m = \left(\frac{\sin}{\cos}\right)^{-1} m.$$

If we have

$$\sec x = \sqrt{1 + \tan^2 x},$$

then $$x = \sec^{-1} \sqrt{1 + \tan^2 x};$$

or, putting $$a = \tan x,$$

$$x = \sec^{-1} \sqrt{1 + a^2}.$$

If we have

$$\cos 2x = 2 \cos^2 x - 1,$$

then $$2x = \cos^{-1}(2 \cos^2 x - 1);$$

or, putting $$\cos x = a,$$

whence $$x = \cos^{-1} a,$$

$$2 \cos^{-1} a = \cos^{-1}(2a^2 - 1).$$

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[1] The beginner may need to guard against the error of supposing that $\sin^{-1} x$ is equivalent to $\frac{1}{\sin x}$. The inverse notation does not mean at all what the algebraic theory of negative exponents would make it.

§ 30. EXERCISES.

1. **Read:**

$$\tan^{-1}\tfrac{1}{3},\ \tan^{-1}(-\tfrac{1}{3}),\ \sin^{-1}\tfrac{3}{4},$$

$$\csc^{-1}\frac{1}{x},\ y = \tan^{-1}\tfrac{6}{5},\ x = \sin^{-1} a,$$

$$\tan^{-1}\tfrac{16}{17} = \tan^{-1}\tfrac{13}{7} - \tan^{-1}\tfrac{1}{3}.$$

2. **Determine the corresponding inverse functions:**

$$\sin 30^\circ = \tfrac{1}{2},\ \tan 45^\circ = 1,$$

$$-y = \cos x,\ \csc^2 a = 1 + \cot^2 a,$$

$$\sin 2x = 2\sin x \cos x,$$

$$\sin 3x = 3\sin x - 4\sin^3 x.$$

3. **Express by the direct notation (without the inverse):**

$$\sin^{-1} y - \csc^{-1}\frac{1}{y} = \cos^{-1}\sqrt{1-y^2},$$

$$\sin^{-1}\tan 45^\circ,\ \tan\sin^{-1}\frac{1}{\sqrt{2}}.$$

4. Construct:

$$\tan^{-1}\tfrac{3}{4},\ \tan^{-1}(-2),\ x = \tan^{-1}(-3),$$

$$\cos^{-1}(-\tfrac{1}{3}),\ y = \sec(-2).$$

5. Prove $\sin^{-1}\frac{\sqrt{3}}{2} = \cos^{-1}\tfrac{1}{2}.$

6. Prove $\sin^{-1}\tfrac{1}{2} + \sin^{-1}\frac{\sqrt{3}}{2} = 90^\circ.$

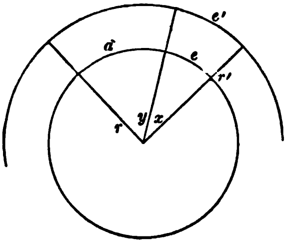
7. Given $\sin^{-1} x + \sin^{-1}\tfrac{2}{3}x = 60^\circ$, to find x.

Ans. $x = \dfrac{3\sqrt{3}}{2\sqrt{19}}.$

§ 31. CIRCULAR MEASURE.

THEOREM I.

Any angle may be measured by the ratio of its intercepted arc to the radius of the circle described from its vertex as a centre.



Since central angles are proportional to their intercepted arcs,

(1) $x : y :: c : a$.

$\therefore$ (2) $x : y :: \frac{c}{r} : \frac{a}{r}$.

Since similar arcs are proportional to their radii,

(3) $c : c' :: r : r'$.

$\therefore$ (4) $\frac{c}{r} = \frac{c'}{r'}$; that is, the ratio of the intercepted arc to the radius *is always the same for the same angle*, whatever be the length of the radius, and hence may be said to determine the angle, or be taken as its measure. Since any quantity may be put equal to its measure, we have, from (2),

$$x = \frac{c}{r}.$$

$$\therefore y = \frac{a}{r}.$$

Scholium. — The ratio of the intercepted arc to the radius is called the *circular measure* of the angle. Hence the definition: *The circular measure of an angle is the ratio of the subtended arc to the radius of a circle described from the vertex of the angle as a centre.*

THEOREM II.

In a circle whose radius is unity, the length of an arc of any number of degrees, minutes, and seconds, is equal to $\frac{n\pi}{180}$,[1] *n being any number, integral or fractional.*

Denoting the circumference by C, and its radius by r,

$$\begin{aligned} C &= 2\pi r \\ &= 2\pi, \text{ when } r = 1; \\ \text{that is, } 360^\circ &= 2\pi. \\ \therefore \quad 1^\circ &= \frac{\pi}{180}; \\ \therefore \quad 1' &= \frac{\pi}{60 \times 180}; \\ \therefore \quad 1'' &= \frac{\pi}{60 \times 60 \times 180}. \end{aligned}$$

. . . .

$$\therefore \text{ arc } a = n^\circ = \frac{n\pi}{180}.$$

Cor. — If x denote the angle, a the intercepted arc described from its vertex as a centre, and r the radius, we shall have, for the circular measure of x,

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[1] The Greek letter π denotes the ratio of the circumference to its diameter, or the length of a semi-circumference whose radius is 1. Its approximate value = 3.14159. The expressions 2π, π, $\frac{1}{2}\pi$, and $\frac{1}{4}\pi$, are often used to denote the angles which they measure, 360°, 180°, 90°, and 45°, respectively.

$$x = \frac{a}{r}. \quad . \quad . \quad . \quad . \quad \textit{Th. I, Scholium.}$$

If $r = 1$, $\quad x = a$;

that is, the circular measure of an angle is equal to *the length of the intercepted arc.*

But, by theorem,

$$a = \frac{n\pi}{180}, \text{ when } r = 1.$$

$$\therefore x = a = \frac{n\pi}{180};$$

that is, *the circular measure of an angle is equal to its number of degrees, minutes, and seconds, multiplied by* $\frac{\pi}{180}$.

§ 32. ANGULAR UNIT OF CIRCULAR MEASURE.

In the equation

$$x = \frac{a}{r},$$

let $a = r$; then

$$x = \frac{r}{r}, = 1.$$

Hence, the angular unit of circular measure is *a central angle whose intercepted arc is equal to the radius of the circle.*

Since central angles are proportional to their intercepted arcs,

$$\frac{x}{y} = \frac{a}{a'}.$$

Let x = angular unit, and y = 4 right angles; then a' = circumference, and

$$\frac{x}{360^\circ} = \frac{a}{2\pi r} = (\text{when } a = r)\ \frac{r}{2\pi r} = \frac{1}{2\pi}.$$

$\therefore$ angular unit $= x$

$$= \frac{360^\circ}{2\pi} = \frac{180^\circ}{\pi} = \frac{180^\circ}{3.14159} = 57^\circ.295779$$

$$= 57^\circ\ 17'\ 44.8''.$$

§ 33. APPLICATION.

In this system of measuring angles,

$$360^\circ = 360 \times \frac{\pi}{180} = 6.2831854.$$

$$180^\circ = 180 \times \frac{\pi}{180} = 3.1415927.$$

$$90^\circ = 90 \times \frac{\pi}{180} = 1.5707963.$$

$$45^\circ = 45 \times \frac{\pi}{180} = .7853982.$$

$$60^\circ = 60 \times \frac{\pi}{180} = 1.0471976.$$

$$30^\circ = 30 \times \frac{\pi}{180} = .5235988.$$

$$1^\circ = 1 \times \frac{\pi}{180} = .01745329.$$

$$1' = \tfrac{1}{60} \times \frac{\pi}{180} = .000290888.$$

$$1'' = \tfrac{1}{3600} \times \frac{\pi}{180} = .00000485.$$

From (1) $x = \frac{n\pi}{180}$, where n denotes *degrees*,

we get (2) $n = x\,\frac{180}{\pi} = x \times 57.295779.$

Hence, *to find the circular measure*, having given the degrees, minutes, and seconds, use formula (1); *to find the degree measure*, having given the circular measure, use formula (2).

Since the circular measure of an angle, to *any* radius, is denoted by the fraction $\frac{arc}{radius} = \frac{a}{r}$ (Th. I, Sch.), from (2) we get

$$(3)\ n = \frac{a}{r} \times \text{angular unit}$$

$$= \frac{a}{r} \times 57^\circ\ 17'\ 44.8''.$$

§ 34. EXERCISES.

1. What is the circular measure of $\frac{5}{8}$ of a right angle?

2. What is the circular measure of $11^\circ\ 15' - \frac{\pi}{16}$?

3. What is the circular measure of $7^\circ\ 30'$?

4. What is the circular measure of $-18^\circ\ 7'$?

5. What is the circular measure of 396°?

6. In a circle whose radius is unity, find the lengths of the following arcs: 60°, 75°, 120°, 225°.

7. Find the lengths of the preceding arcs in a circle whose radius is 10.

Suggestion. — Circumference whose radius is $r = 2\pi r$ = circumference whose radius is $1 \times r$; hence any arc whose radius is r = the corresponding arc to the radius 1 multiplied by the given radius.

8. If the earth's equatorial radius is 3963 miles, what is the length of an arc of 1° on the equator?

9. What is the number of degrees in the angle whose circular measure is $\frac{\pi}{14}$? *Ans.* $12° 51' 25.7''$.

10. What is the number of degrees in the angle whose circular measure is $\frac{5}{2}$ (= $\frac{5}{2}$ of angular unit)?

11. What is the number of degrees in the angle whose circular measure is $-\frac{\pi}{8}$?

12. What is the number of degrees in the central angle whose intercepted arc is 7 feet long, in a circle whose radius is 8 feet?

Suggestion. — Degrees in angle = degrees in arc = circular measure × angular unit $= \frac{arc}{radius} \times 57° 17' 44.8'' = \frac{7}{8} \times 57° 17' 44.8''$.

13. What is the number of degrees in the angle whose intercepted arc is 40 feet, in a circle whose radius is 35 feet?

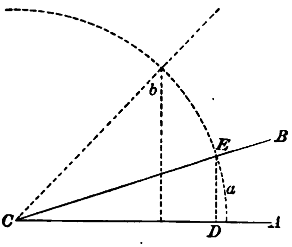
14. The circular measure of the difference between the acute angles of a right triangle is $\frac{\pi}{18}$; how many degrees in each angle?

§ 35. COMPUTATION OF NATURAL FUNCTIONS.

It is required to show only how to construct a complete table of sines for the various angles of the first quadrant, since the sine of an angle is equal to the cosine of its complement; and, having determined the sine and cosine for the first quadrant, the other functions for this and the remaining quadrants are readily computed by formulas previously established.

[NOTE. — It is well for the student to remember continually that the radius of the circle employed in measuring angles is *unity*, — such a value being assumed as will most simplify calculations.]

Let ACB be a very small angle, and CDE its triangle of reference. From C as a centre, with radius = hypothenuse, describe an arc.



Then, $\sin C = \frac{ED}{CE}$.

It is plain that ED, the perpendicular, differs less from the arc a as the angle C decreases. If, then, the angle C were taken small enough, there would be no appreciable difference between the perpendicular and its subtended arc; that is,

$$\frac{ED}{CE} = \frac{a}{CE}, \text{ approximately.}$$

If $C = 1'$,

$$\begin{aligned} \sin C = \sin 1' &= \frac{ED}{CE} = \frac{a}{EC}, \text{ approximately,} \\ &= \textit{circular measure} \text{ of } 1' \\ &= .000290888. \end{aligned}$$

$$\cos 1' = \sqrt{1 - \sin^2 1'} = .999999957.$$

Adding the formula for the sine of the sum to that for the sine of the difference, then the formula for the cosine of the sum to that for the cosine of the difference, and transposing, we have

$$\sin(x + y) = 2 \sin x \cos y - \sin(x - y),$$
$$\cos(x + y) = 2 \cos x \cos y - \cos(x - y).$$

Letting $y = 1'$ *constantly*,

and $x = 1', 2', 3'$, etc., in succession, we have

$$
\begin{aligned}
\sin 2' &= 2\cos 1' \sin 1' - \sin 0' = .0005817764.\\
\sin 3' &= 2\cos 1' \sin 2' - \sin 1' = .0008726646.\\
\sin 4' &= 2\cos 1' \sin 3' - \sin 2' = .0011635526.
\end{aligned}
$$

. . . .

$$
\begin{aligned}
\cos 2' &= 2\cos 1' \cos 1' - \cos 0' = .9999998308.\\
\cos 3' &= 2\cos 1' \cos 2' - \cos 1' = .9999996193.\\
\cos 4' &= 2\cos 1' \cos 3' - \cos 2' = .9999993232.
\end{aligned}
$$

. . . .

If in the formulas, $\sin(x+y) + \sin(x-y) = 2\sin x \cos y$,
and $\cos(x+y) + \cos(x-y) = 2\cos x \cos y$,
we put $x = 30°$, we have

$$
\begin{aligned}
\sin(30° + y) &= \cos y - \sin(30° - y),\\
\cos(30° + y) &= \cos(30° - y) - \sin y.
\end{aligned}
$$

Hence, we may continue the table above 30° by the simple subtraction of the sines and cosines of angles under 30° previously determined.

Thus, making $y = 1', 2', 3'$, etc., successively,

$$
\begin{aligned}
\sin 30°\ 1' &= \cos 1' - 29°\ 49'.\\
\sin 30°\ 2' &= \cos 2' - 29°\ 58'.
\end{aligned}
$$

. . . .

$$
\begin{aligned}
\cos 30°\ 1' &= \cos 29°\ 59' - \sin 1'.\\
\cos 30°\ 2' &= \cos 29°\ 58' - \sin 2'.
\end{aligned}
$$

. . . .

This last process continued to 45°, the sines and cosines of the angles above 45° may be obtained by taking the cosines and sines of their complements below 45°.

The sines and cosines being computed, the remaining functions may be obtained by the formulas,

$$\tan x = \frac{\sin x}{\cos x}, \qquad \cot x = \frac{\cos x}{\sin x} \text{ or } \frac{1}{\tan x},$$

$$\sec x = \frac{1}{\cos x}, \qquad \csc x = \frac{1}{\sin x},$$

$$\text{vers } x = 1 - \cos x, \qquad \text{cvs } x = 1 - \sin x.$$

The tables are not extended beyond 90°, since the functions pass through all their possible numerical values, while the angle varies from zero to a right angle.

Or, for the functions of an angle $> 90°$ and $< 180°$, we may take the *same* functions of the supplement, observing to prefix the proper algebraic sign.

For the functions of an angle $> 180°$ and $< 270°$, we may deduct 180°, and take the same functions of the remainder, prefixing the signs that belong to the third quadrant.

§ 36. INTERPOLATION.

In computing the natural functions, the initial angle is generally taken as 1′, and the table so constructed contains the numerical values of the functions to *degrees* and *minutes*. For functions to degrees, minutes, and *seconds*, interpolations are made similar to those required in using the tables of logarithms of numbers, where the variations of the logarithm are known to be approximately proportional to those of the number. The validity of this principle, as applied to the trigonometric functions, appears as follows:

$$\sin (x + y) = \sin x \cos y + \cos x \sin y,$$

$$\cos (x + y) = \cos x \cos y + \sin x \sin y.$$

If y is very small, we can take

$$\cos y = 1, \text{ and } \sin y = y, \text{ radius being unity;}$$

whence

$$\sin (x + y) = \sin x + y \cos x, \text{ approximately,}$$

and

$$\cos (x + y) = \cos x - y \sin x, \text{ approximately.}$$

$\therefore$ if z is very small,

$$\frac{\sin (x + y) - \sin x}{\sin (x + z) - \sin x} = \frac{y \cos x}{z \cos x} = \frac{y}{z},$$

and

$$\frac{\cos x - \cos (x + y)}{\cos x - \cos (x + z)} = \frac{y \sin x}{z \sin x} = \frac{y}{z}, \text{ wherein}$$

y and z represent the differences of the angles x, $x + y$, and x, $x + z$.

$$\therefore \sin (x + y) - \sin x : \sin (x + z) - \sin x :: y : z;$$

that is, for small variations of an angle, *the differences of the sines, etc., are proportional to the differences of the angles.*

For the utmost accuracy, however, the tables should be constructed to every second, or to every 10″, and the functions given to at least seven decimal places.

Scholium 1. — By inspecting the values of the functions as computed above, it will be seen that, when the angles are very small, the cosines differ from each other very little. Similarly, when the angles are near 90°, the sines differ from each other very little. *If, therefore, the required angle is small, it is better to find it by its sine; if near 90°, by its cosine; or in either case by its tangent.*

Scholium 2. — We have seen [§ 17, *Ex.* 3, (4)] that, within the limits of 0° and 90°, an increase of the angle gives an increase of the sine and tangent, but a *decrease* of the cosine and cotangent. Assuming an increase of the angle, the student will perceive why the interpolation, or tabular difference, is *added* in the case of the sine and tangent, and *subtracted* in that of the cosine and cotangent.

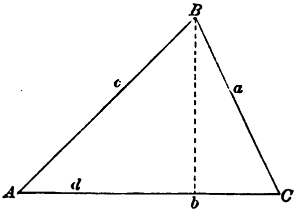
§ 37. PROPERTIES OF TRIANGLES.

THEOREM I.

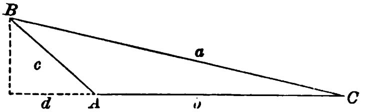
The square of any side of a triangle is equal to the sum of the squares of the other two sides, minus twice the product of these sides into the cosine of their included angle.

If the included angle, as A, is *acute*,

$$a^2 = b^2 + c^2 - 2bd.$$



If the included angle is *obtuse*,



$$a^2 = b^2 + c^2 + 2bd, \text{ by geometry.}$$

But, in the first case,

$$d = c \cos A;$$

and in the second,

$$d = c \cos (180° - A).$$

By substitution,

$$a^2 = b^2 + c^2 - 2bc \cos A,$$

and

$$a^2 = b^2 + c^2 + 2bc \cos (180° - A)$$
$$= b^2 + c^2 + 2bc (- \cos A)$$
$$= b^2 + c^2 - 2bc \cos A \quad . \quad . \quad \S\ 18, (c).$$

Cor. — *The cosine of any angle of a triangle is equal to the sum of the squares of the including sides, minus the square of the opposite side, divided by twice the product of the including sides.*

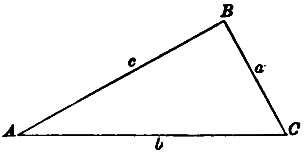
For, from the preceding equation,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Scholium. — The sides of a triangle being given, we are thus able to find the natural cosine of its angles, whence, from a table of natural functions, the angles themselves can be found.

THEOREM II.

The sine of half an angle of a triangle is equal to $\sqrt{\frac{(\frac{1}{2}s - b)(\frac{1}{2}s - c)}{bc}}$, *s representing the sum of the sides, b and c the sides adjacent to the angle.*



For,

$$(1) \qquad \cos A = \frac{b^2 + c^2 - a^2}{2\,b\,c} \quad . \quad . \quad \textit{Th. I, Cor.}$$

and

$$(2) \qquad 1 = 1.$$

$$(2) - (1) = (3)\ 1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2\,b\,c}$$

$$= \frac{2\,b\,c - b^2 - c^2 + a^2}{2\,b\,c}$$

$$= \frac{a^2 - (b - c)^2}{2\,b\,c}$$

$$= \frac{(a + b - c)\ (a + c - b)}{2\,b\,c}.$$

Let $s = a + b + c;$

then $a + b - c = s - 2\,c,$

and $a + c - b = s - 2\,b.$

By substitution,

$$(3) = (4)\ 1 - \cos A = \frac{(s - 2\,b)\ (s - 2\,c)}{2\,b\,c}.$$

But $1 - \cos A = 2 \sin^2 \frac{1}{2} A$ . . . § 24, (5).

$$\therefore\ 2 \sin^2 \tfrac{1}{2} A = \frac{(s - 2\,b)\ (s - 2\,c)}{2\,b\,c}.$$

$$\therefore\ \sin^2 \tfrac{1}{2} A = \frac{(s - 2\,b)\ (s - 2\,c)}{4\,b\,c}$$

$$= \frac{s - 2\,b}{2\,b} \times \frac{s - 2\,c}{2\,c}$$

$$= \frac{\frac{1}{2} s - b}{b} \times \frac{\frac{1}{2} s - c}{c}$$

$$= \frac{(\frac{1}{2} s - b)\ (\frac{1}{2} s - c)}{b\,c}.$$

$$\therefore\ \sin \tfrac{1}{2} A = \sqrt{\frac{(\frac{1}{2} s - b)\ (\frac{1}{2} s - c)}{b\,c}}.$$

THEOREM III.

The cosine of half an angle of a triangle is equal to $\sqrt{\frac{\frac{1}{2}s\left(\frac{1}{2}s-a\right)}{bc}}$, *s representing the sum of the sides, a the side opposite the angle, b and c the sides adjacent.*

For,

$$(1) \qquad \cos A = \frac{b^2+c^2-a^2}{2bc},$$

and

$$(2) \qquad 1 = 1.$$

$$(2)+(1)=(3)\ 1+\cos A = \frac{(b+c)^2-a^2}{2bc}$$

$$= \frac{(a+b+c)(b+c-a)}{2bc}$$

$$= \frac{s(s-2a)}{2bc}.$$

But

$$1+\cos A = 2\cos^2\tfrac{1}{2}A \quad . \quad . \quad . \quad \S\ 24,\ (6).$$

$$\therefore 2\cos^2\tfrac{1}{2}A = \frac{s(s-2a)}{2bc}.$$

$$\therefore \quad \cos^2\tfrac{1}{2}A = \frac{s(s-2a)}{4bc} = \frac{\frac{1}{2}s\left(\frac{1}{2}s-a\right)}{bc}.$$

$$\therefore \quad \cos\tfrac{1}{2}A = \sqrt{\frac{\frac{1}{2}s\left(\frac{1}{2}s-a\right)}{bc}}.$$

THEOREM IV.

The tangent of half an angle of a triangle is equal to $\sqrt{\frac{\left(\frac{1}{2}s-b\right)\left(\frac{1}{2}s-c\right)}{\frac{1}{2}s\left(\frac{1}{2}s-a\right)}}$, *s representing the sum of the sides, b and c the sides adjacent to the angle, and a the side opposite.*

For,

$$(1)\ \sin \tfrac{1}{2} A = \sqrt{\frac{(\frac{1}{2} s - b)\ (\frac{1}{2} s - c)}{b\, c}} \quad \textit{Th. II,}$$

and

$$(2)\ \cos \tfrac{1}{2} A = \sqrt{\frac{\frac{1}{2} s\ (\frac{1}{2} s - a)}{b\, c}} \quad . \ . \quad \textit{Th. III.}$$

$$(1) \div (2) = (3)\ \tan \tfrac{1}{2} A = \sqrt{\frac{(\frac{1}{2} s - b)\ (\frac{1}{2} s - c)}{\frac{1}{2} s\ (\frac{1}{2} s - a)}}.$$

Scholium 1. — The formulae of Theorems II, III, and IV, unlike that of Theorem I, Cor., are adapted to convenient computation by logarithms. Thus,

$$\text{L.} \tan \tfrac{1}{2} A = 10 + \frac{\log(\frac{1}{2} s - b) + \log(\frac{1}{2} s - c) - \log \frac{1}{2} s - \log(\frac{1}{2} s - a)}{2}$$

$$= \frac{\log(\frac{1}{2} s - b) + \log(\frac{1}{2} s - c) + 10 - \log \frac{1}{2} s + 10 - \log(\frac{1}{2} s - a)}{2}$$

$$= \frac{\log(\frac{1}{2} s - b) + \log(\frac{1}{2} s - c)}{2} + \frac{a.\,c. \log \frac{1}{2} s + a.\,c. \log(\frac{1}{2} s - a)}{2}.$$

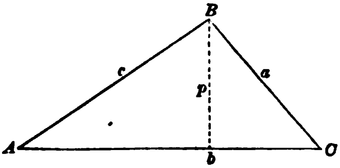
Scholium 2. — If the half-angle is less than 45°, the formula for the sine is preferred; if greater than 45°, the formula for the cosine . . . . . . § 35, *Scholium 1.*

If all the angles are required, it is better to use the formula for the tangent; for, in addition to being accurate for all values of the angle, it requires only the logarithms of $\frac{1}{2} s$, $\frac{1}{2} s - a$, $\frac{1}{2} s - b$, $\frac{1}{2} s - c$.

THEOREM V.

The area of a triangle is equal to half the product of any two sides multiplied by the sine of their included angle.

Let T denote the area of the triangle ABC; then



$$T = \tfrac{1}{2} bp.$$

But $$p = c \sin A;$$

$$\therefore \quad T = \tfrac{1}{2} bc \sin A.$$

Similarly, $$T = \tfrac{1}{2} ab \sin C,$$

and $$T = \tfrac{1}{2} ac \sin B.$$

From which the area may be determined, when any two sides and their included angle are given.

THEOREM VI.

The area of a triangle is equal to

$$\sqrt{\tfrac{1}{2} s \left(\tfrac{1}{2} s - a\right) \left(\tfrac{1}{2} s - b\right) \left(\tfrac{1}{2} s - c\right)},$$

a, b, and c representing the sides, and s their sum.

$$T = \tfrac{1}{2} bc \sin A.$$

But $$\sin A = 2 \sin \tfrac{1}{2} A \cos \tfrac{1}{2} A,$$

and $$\sin \tfrac{1}{2} A = \sqrt{\frac{\left(\tfrac{1}{2} s - b\right)\left(\tfrac{1}{2} s - c\right)}{bc}},$$

and $$\cos \tfrac{1}{2} A = \sqrt{\frac{\tfrac{1}{2} s \left(\tfrac{1}{2} s - a\right)}{b c}}.$$

$$\therefore \qquad \sin A = \frac{2 \sqrt{\tfrac{1}{2} s \left(\tfrac{1}{2} s - a\right) \left(\tfrac{1}{2} s - b\right) \left(\tfrac{1}{2} s - c\right)}}{b c}.$$

$$\therefore \qquad T = \sqrt{\tfrac{1}{2} s \left(\tfrac{1}{2} s - a\right) \left(\tfrac{1}{2} s - b\right) \left(\tfrac{1}{2} s - c\right)}.$$

From which the area of a triangle may be determined, when the three sides are given.

Cor. — *The sine of any angle of a triangle is equal to* $\frac{2 \sqrt{\tfrac{1}{2} s \left(\tfrac{1}{2} s - a\right) \left(\tfrac{1}{2} s - b\right) \left(\tfrac{1}{2} s - c\right)}}{b c}$, *$a$, b, and c representing the sides, s their sum, b and c the sides adjacent to the angle.*

THEOREM VII.

Any side of a triangle is equal to the product of half the perimeter, by the sine of the opposite angle, divided by the product of the cosines of the adjacent half-angles.

For,

$$(1) \qquad \frac{b}{a} = \frac{\sin B}{\sin A},$$

and $$(2) \qquad \frac{c}{a} = \frac{\sin C}{\sin A} \quad \ldots \quad \textit{Ch. II}, \S\ 31.$$

$$(1) + (2) = (3) \qquad \frac{b + c}{a} = \frac{\sin B + \sin C}{\sin A}$$

$$= \frac{\sin B + \sin C}{\sin (B + C)} \quad . \ . \quad \S\ 18,\ (c).$$

$$= \frac{\cos \tfrac{1}{2} (B - C)}{\cos \tfrac{1}{2} (B + C)} \quad . \ . \quad \S\ 26,\ (16).$$

$$(4) \qquad 1 = 1.$$

$$(3) + (4) = (5)\ \frac{a+b+c}{a} = \frac{\cos\frac{1}{2}(B+C) + \cos\frac{1}{2}(B-C)}{\cos\frac{1}{2}(B+C)}$$

$$= \frac{2\cos\frac{1}{2}B\cos\frac{1}{2}C}{\sin\frac{1}{2}A}.\ \S\,26,\ (3)-(7).$$

Let $p = a + b + c$,

then $a = \dfrac{\frac{1}{2}p\sin\frac{1}{2}A}{\cos\frac{1}{2}B\cos\frac{1}{2}C}$.

From which the sides may be determined, when the perimeter and the angles are given.

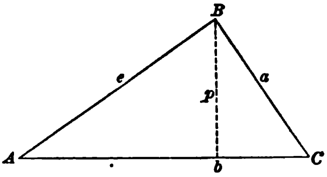
THEOREM VIII.

The perpendicular from any vertex of a triangle to the opposite side is equal to the product of that side by the product of the sines of the adjacent angles, divided by the sine of the sum of those angles.

For,

$$p = a \sin C,$$

and, $\sin A : \sin B :: a : b$;



$$\therefore \qquad a = \frac{b\sin A}{\sin B};$$

$$\therefore \qquad p = \frac{b\sin A\sin C}{\sin B} = \frac{b\sin A\sin C}{\sin(A+C)},$$ since

$A + C =$ supplement of B.

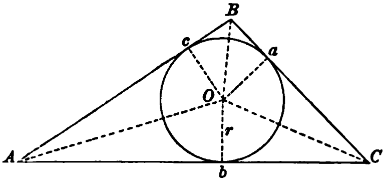
PROBLEM I.

Given the three sides of a triangle, to find the radius of the inscribed circle.

$$AOB = \tfrac{1}{2}\,a\,r,$$

$$AOC = \tfrac{1}{2}\,b\,r,$$

$$AOB = \tfrac{1}{2}\,c\,r.$$



$$\therefore\quad ABC = BOC + AOC + AOB = \tfrac{1}{2}r\,(a+b+c) = \tfrac{1}{2}rs.$$

$$\text{But } ABC = \sqrt{\tfrac{1}{2}s\,(\tfrac{1}{2}s-a)\,(\tfrac{1}{2}s-b)\,(\tfrac{1}{2}s-c)}\quad .\ .\quad \textit{Th. VI.}$$

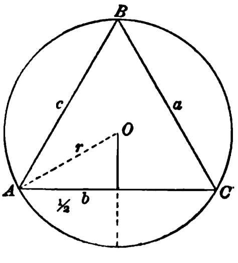
$$\therefore\quad \tfrac{1}{2}rs = \sqrt{\tfrac{1}{2}s\,(\tfrac{1}{2}s-a)\,(\tfrac{1}{2}s-b)\,(\tfrac{1}{2}s-c)};$$

$$\therefore\quad r = \frac{\sqrt{\tfrac{1}{2}s\,(\tfrac{1}{2}s-a)\,(\tfrac{1}{2}s-b)\,(\tfrac{1}{2}s-c)}}{\tfrac{1}{2}s} = \frac{T}{\tfrac{1}{2}s}$$

$$= \sqrt{\frac{(\tfrac{1}{2}s-a)\,(\tfrac{1}{2}s-b)\,(\tfrac{1}{2}s-c)}{\tfrac{1}{2}s}}.$$

PROBLEM II.

Given the three sides of a triangle, to find the radius of the circumscribed circle.



Let ABC be a triangle whose three sides, a, b, c, are given; and let O be the centre of the circumscribed circle. From O let fall a perpendicular upon b.

$$\text{Angle } O = \text{angle } B,$$

$$\text{since each} = \tfrac{1}{2}\,\underset{\smile}{AC}.$$

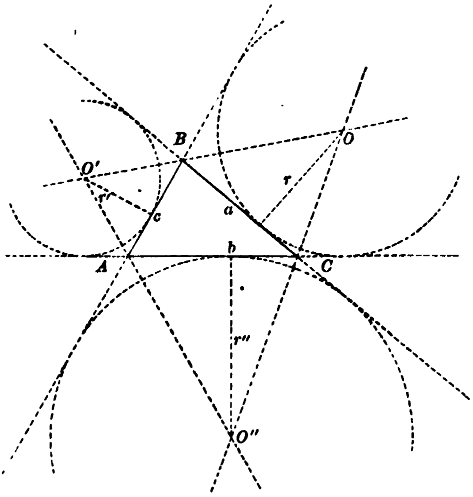
$$\tfrac{1}{2}\,b = r \sin O = r \sin B;$$

$$\therefore r = \frac{b}{2 \sin B} = \frac{b}{\dfrac{4\sqrt{\frac{1}{2}s\,(\frac{1}{2}s-a)\,(\frac{1}{2}s-b)\,(\frac{1}{2}s-c)}}{a\,c}}$$

$$= \frac{b}{\dfrac{4\,T}{a\,c}} = \frac{a\,b\,c}{4\,T} \;\;.\;.\;.\;.\;.\;.\;. \quad \textit{Th. VI, Cor.}$$

PROBLEM III.

Given the three sides of a triangle, to find the radii of the escribed circles.



Escribed circles are those exterior to the triangle, each touching one side of the triangle and the other two sides produced.

Their centres are the points of intersection of the bisectors of the exterior angles.

Their radii will be perpendiculars to the given sides.

Let ABC be a triangle whose sides, a, b, c, are given; and O, O', O'', the centres of the escribed circles whose radii, r, r', r'', are required.

By Theorem VIII,

$$r = \frac{a \sin \frac{1}{2}(180^\circ - B) \sin \frac{1}{2}(180^\circ - C)}{\sin \frac{1}{2}[360^\circ - (B + C)]}$$

$$= \frac{a \sin (90^\circ - \frac{1}{2} B) \sin (90^\circ - \frac{1}{2} C)}{\sin [180^\circ - \frac{1}{2}(B + C)}$$

$$= \frac{a \cos \frac{1}{2} B \cos \frac{1}{2} C}{\sin \frac{1}{2}(B + C)} = \frac{a \cos \frac{1}{2} B \cos \frac{1}{2} C}{\cos \frac{1}{2} A} = \frac{1}{2} p \tan \frac{1}{2} A;$$

$$r' = \frac{c \cos \frac{1}{2} A \cos \frac{1}{2} B}{\cos \frac{1}{2} C} = \frac{1}{2} p \tan \frac{1}{2} C;$$

$$r'' = \frac{b \cos \frac{1}{2} A \cos \frac{1}{2} C}{\cos \frac{1}{2} B} = \frac{1}{2} p \tan \frac{1}{2} B \quad . \quad . \quad . \quad . \quad \textit{Th. VII.}$$

$$r = \frac{1}{2} p \sqrt{\frac{(\frac{1}{2} s - b)(\frac{1}{2} s - c)}{\frac{1}{2} s (\frac{1}{2} s - a)}} = \sqrt{\frac{\frac{1}{4} p^2 (\frac{1}{2} s - b)(\frac{1}{2} s - c)}{\frac{1}{2} s (\frac{1}{2} s - a)}}$$

$$= \sqrt{\frac{(\frac{1}{2} s (\frac{1}{2} s - a)(\frac{1}{2} s - b)(\frac{1}{2} s - c)}{(\frac{1}{2} s - a)^2}} = \frac{T}{\frac{1}{2} s - a} \text{ (since } p = s\text{)};$$

$$r' = \sqrt{\frac{\frac{1}{2} s (\frac{1}{2} s - a)(\frac{1}{2} s - b)(\frac{1}{2} s - c)}{(\frac{1}{2} s - c)^2}} = \frac{T}{\frac{1}{2} s - c};$$

$$r'' = \sqrt{\frac{\frac{1}{2} s (\frac{1}{2} s - a)(\frac{1}{2} s - b)(\frac{1}{2} s - c)}{(\frac{1}{2} s - b)^2}} = \frac{T}{\frac{1}{2} s - b}.$$

§ 38. EXERCISES.

1. Given $\left\{\begin{array}{l} a = 10, \\ b = 12, \\ c = 14. \end{array}\right\}$ Required $\left\{\begin{array}{l} A = 44^\circ\ 24'\ 55''.2, \\ B = 57^\circ\ \ 7'\ 18'', \\ C = 78^\circ\ 27'\ 47''. \end{array}\right\}$

2. Given $\left\{\begin{array}{l} a = .8706, \\ b = .0916, \\ c = .7902. \end{array}\right\}$ Required $\left\{\begin{array}{l} A = 149^\circ\ 49'\ 0''.4, \\ B = 3^\circ\ \ 1'\ 56''.2, \\ C = 27^\circ\ \ 9'\ \ 3''.4. \end{array}\right\}$

3. Two sides of a triangle are 55 and 45 respectively, and their included angle is 36° 52′; find the area.

4. The sides of a triangle are 20, 30, and 40; find the area.

5. The perimeter of a triangle is 375, and the angles are 55° 46′ 18″, 82° 49′ 8″, and 41° 24′ 34″, respectively; find the sides.

6. The sides of a triangle are 7, 9, and 10; find the radius of the inscribed circle, that of the circumscribed circle, and the radii of the escribed circles.

CHAPTER IV.

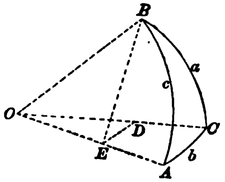
SPHERICAL TRIGONOMETRY.

§ 1. DEFINITIONS.

1. If a sphere be described about the vertex of a triedral angle as a centre, the intersections of the three faces of the triedral with the surface of the sphere are arcs of great circles, and these arcs together constitute a **Spherical Triangle.**

2. To *solve* a spherical triangle is to compute any three of its parts when the other three parts are given.

3. The *sides* are measured in degrees, minutes, and seconds, and therefore by the plane angles formed by radii of the sphere drawn to the vertices of the triangle. Hence their measures are independent of the length of the radius, which may be assumed to have any convenient numerical value, as, for instance, *unity*.



Thus,

$$b = AOC, \quad c = AOB.$$

4. Each *angle* is regarded as identical with that included between the planes of its sides; and its measure is equal to that of the angle formed by two lines, one in each plane, and both perpendicular to the common intersection of the planes at the same point. Thus, the angle A of the triangle ABC represents the inclination of the planes AOB, AOC, whose measure is BED.

5. Hence, **Spherical Trigonometry** treats of the numerical relations between the elements of a spherical triangle; or, which is the same thing, between the six parts of the triedral that intercepts it, — the three face-angles and the three diedrals.

6. In this work only those triangles are considered whose parts are positive, and each less than 180°. The following principles, proved in Geometry, are grouped here for the convenience of the student:

The sum of the three sides lies between 0° and 360°.

The sum of the three angles lies between 180° and 540°.

Each side is less than the sum of the other two.

Each angle is greater than the difference between 180° and the sum of the other two.

Of any two unequal sides, the greater lies opposite the greater angle; and conversely.

Each side or angle is the supplement of the corresponding angle or side of the polar triangle.

If two sides of a triangle are equal, so also are the opposite angles; and conversely.

The perpendicular from the vertex of an isosceles triangle to the base bisects both the vertical angle and the base.

A spherical triangle is, in general, determined when any three of its six parts are known.

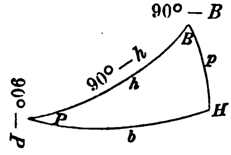
The area is to the hemisphere as the excess of the sum of the angles over 180° is to 360°.

7. Two parts of a spherical triangle are of the **same species** when both are less than 90° or both greater than 90°.

8. Two parts of a spherical triangle are of **different species** when one part is less than 90° and the other part greater than 90°.

9. If, in a right spherical triangle, we consider only the three sides and the two oblique angles, then the two sides which include the right angle, the complements of the oblique angles and of the hypothenuse, are called the **Circular Parts**. Thus, in the triangle HBP, right-angled at H, the circular parts are

$$b, \ p, \ 90° - B, \ 90° - P, \ 90° - h.$$



10. The **Adjacent Parts** of a right triangle are those which are not separated by a circular part: p and b; p and $90° - B$; $90° - B$ and $90° - h$; $90° - h$ and $90° - P$; $90° - P$ and b.

11. The **Opposite Parts** of a right triangle are those which are separated by a circular part: p and $90° - h$; $90° - B$ and $90° - P$; $90° - h$ and b; $90° - P$ and p; b and $90° - B$.

12. A **Middle Part** of a right triangle is that which is adjacent to two other parts, or opposite two other parts.

§ 2. EXERCISES.

1. Give the middle part, and state whether the other parts are adjacent or opposite:

(1.) $90° - h,\quad 90° - B,\quad p.$
(2.) $90° - P,\quad 90° - B,\quad p.$
(3.) $b,\quad 90° - P,\quad p.$
(4.) $90° - B,\quad 90° - h,\quad b.$
(5.) $90° - P,\quad 90° - B,\quad b.$

2. If h and b are given to find B, what are the circular parts to be considered? Which is the middle part? Are the extremes adjacent or opposite?

3. If B and P are given to find b, indicate the circular parts, tell which is middle, and whether the remaining two are adjacent or opposite.

4. What are the opposite parts when p is middle? Adjacent parts?

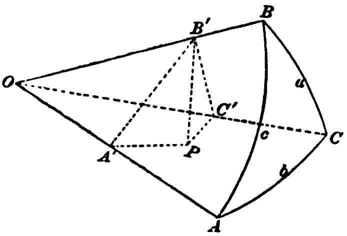
5. State the opposite and the adjacent parts when b is middle. When complement of B is middle.

§ 3. PRINCIPLES.

THEOREM I.

In any spherical triangle, the sines of the sides are proportional to the sines of the opposite angles.

Let ABC be a spherical triangle on a sphere whose centre is O. From any point in OB let fall a perpendicular $B'P$ to the plane AOC. Let $A'B'$, $A'P$, be perpendicular to OA at A'. Let PC' and $B'C'$ be perpendicular to OC at C'. Then the plane triangles $A'PB'$, $B'PC'$, are right-angled at



P; $OA'B'$, $OC'B'$, are right-angled at A' and C'; angle $B'A'P = A$; $B'C'P = C$; $a = B'OC'$; and $c = B'OA'$. Therefore,

$$\sin A = \sin B'A'P = \frac{B'P}{B'A'},$$

$$\sin C = \sin B'C'P = \frac{B'P}{B'C'};$$

$$\frac{\sin A}{\sin C} = \frac{B'P}{B'A'} \times \frac{B'C'}{B'P} = \frac{B'C'}{B'A'}.$$

Also,

$$\sin a = \sin B'OC' = \frac{B'C'}{B'O},$$

$$\sin c = \sin B'OA' = \frac{B'A'}{B'O};$$

$$\therefore \frac{\sin a}{\sin c} = \frac{B'C'}{B'A'}.$$

$$\therefore \frac{\sin a}{\sin c} = \frac{\sin A}{\sin C}, \text{ whence}$$

$$\sin a : \sin c :: \sin A : \sin C.$$

Similarly,

$$\sin a : \sin b :: \sin A : \sin B,$$

and

$$\sin b : \sin c :: \sin B : \sin C.$$

Cor. I.— *The preceding proportions give:*

(1) $\sin a \sin C = \sin c \sin A$.

(2) $\sin a \sin B = \sin b \sin A$.

(3) $\sin b \sin C = \sin c \sin B$.

Cor. II.— *The sine of either oblique angle of a right triangle is equal to the quotient of the sine of the opposite side divided by the sine of the hypothenuse.*

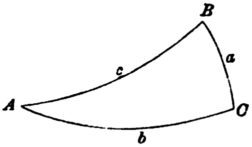
For,

$$\sin A = \frac{\sin a}{\sin c} \sin C.$$

If, $$C = 90^\circ,$$

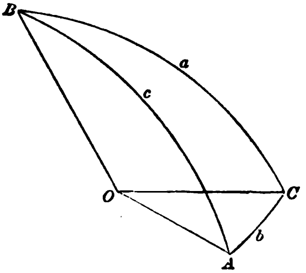
$$\sin C = 1.$$

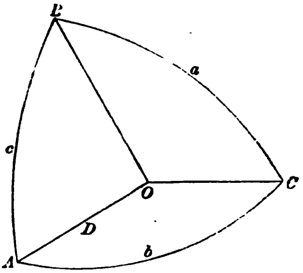
$$\therefore \sin A = \frac{\sin a}{\sin c}.$$



Similarly, $$\sin B = \frac{\sin b}{\sin c}.$$

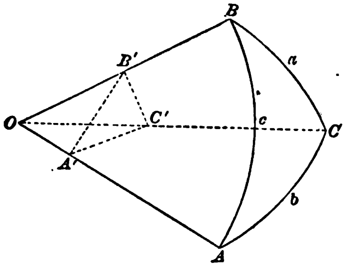
Scholium. — It has been assumed that the parts are each less than 90°. The construction, however, would not vary, if any of the parts were greater than 90°, except that the points A' and C' might be found in AO, CO, produced through O; and one or more of the right triangles would contain the supplements of A, a, C, c, etc. But the preceding demonstration would still be valid, since the sine of an angle is the same as the sine of its supplement.





THEOREM II.

In any spherical triangle, the cosine of any side is equal to the product of the cosines of the other two sides, plus the product of their sines into the cosine of their included angle.



Let ABC be a spherical triangle, and O the centre of the sphere. Let a plane, perpendicular to AO, intersect the faces of the triedral angle in the lines $A'B'$, $B'C'$, $A'C'$. Then

$$B'A'C' = A,$$

$$B'OC' = a;$$ and in the triangles $A'B'C'$, $OB'C'$,

$$(1)\quad \overline{B'C'}^2 = \overline{A'B'}^2 + \overline{A'C'}^2 - 2\,A'B' \times A'C' \cos A,$$

$$(2)\quad \overline{B'C'}^2 = \overline{OB'}^2 + \overline{OC'}^2 - 2\,OB' \times OC' \cos a.$$

$$(2)-(1)=(3)\qquad 0 = \overline{OB'}^2 - \overline{A'B'}^2 + \overline{OC'}^2 - \overline{A'C'}^2 + 2\,A'B' \times A'C' \cos A - 2\,OB' \times OC' \cos a.$$

But $$\overline{OA'}^2 = \overline{OB'}^2 - \overline{A'B'}^2,$$

and $$\overline{OA'}^2 = \overline{OC'}^2 - \overline{A'C'}^2.$$

$$\therefore\quad 0 = 2\,\overline{OA'}^2 + 2\,A'B' \times A'C' \cos A - 2\,OB' \times OC' \cos a.$$

$$\therefore \cos a = \frac{OA'}{OB'} \times \frac{OA'}{OC'} + \frac{A'B'}{OB'} \times \frac{A'C'}{OC'} \cos A.$$

But $$\frac{OA'}{OB'} \times \frac{OA'}{OC'} = \cos b \cos c,$$

and $$\frac{A'B'}{OB'} \times \frac{A'C'}{OC'} = \sin b \sin c;$$

$$\therefore \ (1) \ \cos a = \cos b \cos c + \sin b \sin c \cos A.$$

Similarly,

$$(2) \ \cos b = \cos a \cos c + \sin a \sin c \cos B.$$

$$(3) \ \cos c = \cos a \cos b + \sin a \sin b \cos C.$$

Cor. I. — Multiplying (1) by $\cos c$, we have

$$\cos a \cos c = \cos b \cos^2 c + \sin b \sin c \cos c \cos A.$$

From (2), $\cos a \cos c + \sin a \sin c \cos B = \cos b.$

Subtracting and factoring, we have

$\sin a \sin c \cos B = (1 - \cos^2 c) \cos b - \sin b \sin c \cos c \cos A.$

But, $1 - \cos^2 c = \sin^2 c$. Substituting, and dividing by $\sin c$, we have

(x) $\sin a \cos B = \sin c \cos b - \cos c \sin b \cos A$; similarly,

(y) $\sin b \cos C = \sin a \cos c - \cos a \sin c \cos B$; and

(z) $\sin c \cos A = \sin b \cos a - \cos b \sin a \cos C.$

Cor. II. — Dividing these formulas respectively by

$$\frac{\sin a \sin B}{\sin A} = \sin b, \text{ derived from (2), } \textit{Th. I, Cor. I},$$

and reducing, we have

$$(x) \ \sin A \cot B = \sin c \cot b - \cos c \cos A.$$

Similarly,

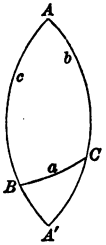
$$(y) \ \sin B \cot C = \sin a \cot c - \cos a \cos B,$$

and $$(z) \ \sin C \cot A = \sin b \cot a - \cos b \cos C.$$

Cor. III. — If we here interchange the letters B and C, and therefore also b and c, we find

$$(x)\quad \sin A \cot C = \sin b \cot c - \cos b \cos A.$$
$$(y)\quad \sin B \cot A = \sin c \cot a - \cos c \cos B.$$
$$(z)\quad \sin C \cot B = \sin a \cot b - \cos a \cos C.$$

Scholium. — Both b and c are supposed less than 90°, while A and a are not restricted. But the preceding formulas are no less applicable to other cases.



Thus, let $b > 90°$, $c > 90°$. Produce AB and AC to meet in A', forming the line AA'. Then $A'B = 180° - c$, and $A'C = 180° - b$, are both less than 90°, and the theorem is applicable to $A'BC$. Hence,

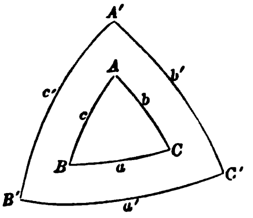
$$\begin{aligned}\cos a &= \cos(180° - b)\cos(180° - c) \\ &\qquad + \sin(180° - b)\sin(180° - c)\cos A, \\ &= (-\cos b)(-\cos c) + \sin b \sin c \cos A, \\ &= \cos b \cos c + \sin b \sin c \cos A,\end{aligned}$$

the same result as would have been obtained by applying the theorem directly to ABC.

THEOREM III.

The cosine of any angle of a spherical triangle is equal to the product of the sines of the other angles into the cosine of their included side, minus the product of the cosines of these angles.

Let ABC be a spherical triangle and $A'B'C'$ its polar. Then, by geometry,



$$a = 180° - A', \quad A = 180° - a',$$
$$b = 180° - B', \quad B = 180° - b',$$
$$c = 180° - C', \quad C = 180° - c'.$$

Substituting these values in the formulas of Theorem II, and reducing,

$$-\cos A' = \cos B' \cos C' - \sin B' \sin C' \cos a'.$$
$$-\cos B' = \cos A' \cos C' - \sin A' \sin C' \cos b'.$$
$$-\cos C' = \cos A' \cos B' - \sin A' \sin B' \cos c'.$$

Changing the signs and omitting the accents, since the formulas are true universally,

(1) $\cos A = \sin B \sin C \cos a - \cos B \cos C.$

(2) $\cos B = \sin A \sin C \cos b - \cos A \cos C.$

(3) $\cos C = \sin A \sin B \cos c - \cos A \cos B.$

THEOREM IV.

The sine of one half of any angle of a spherical triangle is equal to the square root of the quotient obtained by dividing the sine of one half the sum of the sides minus one adjacent side into the sine of one half the sum minus the other adjacent side, by the product of the sines of the adjacent sides.

For,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c},$$

and $$1 - \cos A = \frac{\cos b \cos c + \sin b \sin c - \cos a}{\sin b \sin c}.$$

But $$1 - \cos A = 2 \sin^2 \tfrac{1}{2} A,$$

and

$$\cos b \cos c + \sin b \sin c = \cos (b - c).$$

$$\therefore \quad 2 \sin^2 \tfrac{1}{2} A = \frac{\cos (b - c) - \cos a}{\sin b \sin c}.$$

But $\cos (b - c) - \cos a = 2 \sin \frac{1}{2} (a + c - b) \sin \frac{1}{2} (a + b - c)$.

$$\therefore \quad \sin^2 \tfrac{1}{2} A = \frac{\sin \frac{1}{2} (a + c - b) \sin \frac{1}{2} (a + b - c)}{\sin b \sin c}.$$

$$\therefore \quad (1) \quad \sin \tfrac{1}{2} A = \sqrt{\frac{\sin (\frac{1}{2} s - b) \sin (\frac{1}{2} s - c)}{\sin b \sin c}}.$$

Similarly, $$(2) \quad \sin \tfrac{1}{2} B = \sqrt{\frac{\sin (\frac{1}{2} s - a) \sin (\frac{1}{2} s - c)}{\sin a \sin c}},$$

and $$(3) \quad \sin \tfrac{1}{2} C = \sqrt{\frac{\sin (\frac{1}{2} s - a) \sin (\frac{1}{2} s - b)}{\sin a \sin b}}.$$

THEOREM V.

The cosine of one half of any angle of a spherical triangle is equal to the square root of the quotient obtained by dividing the sine of one half the sum of the sides into the sine of one half the sum minus the side opposite the angle, by the product of the sines of the adjacent sides.

For,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} \quad . \; . \quad Th.\ II,\ (1).$$

Adding 1, $1 + \cos A = \dfrac{\cos a + \sin b \sin c - \cos b \cos c}{\sin b \sin c}$.

But $\quad 1 + \cos A = 2 \cos^2 \frac{1}{2} A,$

and

$\sin b \sin c - \cos b \cos c = - \cos (b + c).$

$$\therefore \quad 2 \cos^2 \tfrac{1}{2} A = \frac{\cos a - \cos (b + c)}{\sin b \sin c}.$$

But

$$\cos a - \cos (b + c) = 2 \sin \tfrac{1}{2} (a + b + c) \sin \tfrac{1}{2} (b + c - a).$$

$$\therefore \quad \cos^2 \tfrac{1}{2} A = \frac{\sin \frac{1}{2} (a + b + c) \sin \frac{1}{2} (b + c - a)}{\sin b \sin c}$$

$$= \frac{\sin \frac{1}{2} s \sin (\frac{1}{2} s - a)}{\sin b \sin c}.$$

$$\therefore \quad (1) \cos \tfrac{1}{2} A = \sqrt{\frac{\sin \frac{1}{2} s \sin (\frac{1}{2} s - a)}{\sin b \sin c}}.$$

Similarly, $(2) \cos \frac{1}{2} B = \sqrt{\dfrac{\sin \frac{1}{2} s \sin (\frac{1}{2} s - b)}{\sin a \sin c}}$,

and $\quad (3) \cos \frac{1}{2} C = \sqrt{\dfrac{\sin \frac{1}{2} s \sin (\frac{1}{2} s - c)}{\sin a \sin b}}$.

THEOREM VI.

The tangent of one half of any angle of a spherical triangle is equal to the square root of the quotient obtained by dividing the sine of one half the sum of the sides minus one adjacent side into the sine of one half the sum minus *the other adjacent side, by the sine of one half the sum of the sides into the sine of one half the sum minus the opposite side.*

For, dividing (1), Theorem V, by (1), Theorem IV, we obtain

$$(1)\quad \tan \tfrac{1}{2} A = \sqrt{\frac{\sin (\frac{1}{2} s - b) \sin (\frac{1}{2} s - c)}{\sin \frac{1}{2} s \sin (\frac{1}{2} s - a)}}.$$

Similarly,

$$(2)\quad \tan \tfrac{1}{2} B = \sqrt{\frac{\sin (\frac{1}{2} s - a) \sin (\frac{1}{2} s - c)}{\sin \frac{1}{2} s \sin (\frac{1}{2} s - b)}},$$

and

$$(3)\quad \tan \tfrac{1}{2} C = \sqrt{\frac{\sin (\frac{1}{2} s - a) \sin (\frac{1}{2} s - b)}{\sin \frac{1}{2} s \sin (\frac{1}{2} s - c)}}.$$

THEOREM VII.

The sine of one half of any side of a spherical triangle is equal to the square root of the quotient obtained by dividing minus the cosine of one half the sum of the angles into the cosine of one half the sum minus the angle opposite the side, by the product of the sines of the adjacent angles.

For, passing to the polar of ABC,

$$a = 180° - A', \quad b = 180° - B', \quad c = 180° - C',$$

$$A = 180° - a', \quad B = 180° - b', \quad C = 180° - c'.$$

$$\tfrac{1}{2} s = 270° - \tfrac{1}{2} (A' + B' + C'),$$

$$\tfrac{1}{2} s - a = 90° - \tfrac{1}{2} (B' + C' - A').$$

Making $S = A' + B' + C'$, substituting in (1) of Theorem IV, reducing, and omitting primes, we have

$$(1)\quad \sin \tfrac{1}{2} a = \sqrt{\frac{-\cos \frac{1}{2} S \cos (\frac{1}{2} S - A)}{\sin B \sin C}}.$$

In like manner,

$$(2)\quad \sin \tfrac{1}{2} b = \sqrt{\frac{-\cos \frac{1}{2} S \cos (\frac{1}{2} S - B)}{\sin A \sin C}},$$

and

$$(3)\quad \sin \tfrac{1}{2} c = \sqrt{\frac{-\cos \frac{1}{2} S \cos (\frac{1}{2} S - C)}{\sin A \sin B}}.$$

THEOREM VIII.

The cosine of one half of any side of a spherical triangle is equal to the square root of the quotient obtained by dividing the cosine of one half the sum of the angles minus one adjacent angle into the cosine of half the sum minus the other adjacent angle, by the product of the sines of the adjacent angles.

For, using the formulas of Theorem V, passing to the polar, making $S = A' + B' + C'$, substituting, reducing, and omitting primes, we find

$$(1)\quad \cos \tfrac{1}{2} a = \sqrt{\frac{\cos (\frac{1}{2} S - B) \cos (\frac{1}{2} S - C)}{\sin B \sin C}}.$$

$$(2)\quad \cos \tfrac{1}{2} b = \sqrt{\frac{\cos (\frac{1}{2} S - A) \cos (\frac{1}{2} S - C)}{\sin A \sin C}}.$$

$$(3)\quad \cos \tfrac{1}{2} c = \sqrt{\frac{\cos (\frac{1}{2} S - A) \cos (\frac{1}{2} S - B)}{\sin A \sin B}}.$$

THEOREM IX.

The tangent of one half of any side of a spherical triangle is equal to the square root of the quotient obtained by dividing minus the cosine of one half the sum of the angles into the cosine of one half the sum minus the angle opposite the side, by the cosine of one half the sum of the angles minus one adjacent angle into the cosine of one half the sum minus the other adjacent angle.

For, dividing (1), (2) and (3) of Theorem VII by (1), (2) and (3) of Theorem VIII, we have

$$(1) \quad \tan \tfrac{1}{2} a = \sqrt{\frac{-\cos \tfrac{1}{2} S \cos (\tfrac{1}{2} S - A)}{\cos (\tfrac{1}{2} S - B) \cos (\tfrac{1}{2} S - C)}}.$$

$$(2) \quad \tan \tfrac{1}{2} b = \sqrt{\frac{-\cos \tfrac{1}{2} S \cos (\tfrac{1}{2} S - B)}{\cos (\tfrac{1}{2} S - A) \cos (\tfrac{1}{2} S - C)}}.$$

$$(3) \quad \tan \tfrac{1}{2} c = \sqrt{\frac{-\cos \tfrac{1}{2} S \cos (\tfrac{1}{2} S - C)}{\cos (\tfrac{1}{2} S - A) \cos (\tfrac{1}{2} S - B)}}.$$

THEOREM X.

In any spherical triangle, the sine of one half the sum of any two angles is to the sine of one half their difference as the tangent of one half the included side is to the tangent of one half the difference of the opposite sides.

For,

$$\frac{\tan \tfrac{1}{2} A}{\tan \tfrac{1}{2} B} = \frac{\sin (\tfrac{1}{2} s - b)}{\sin (\tfrac{1}{2} s - a)} \;.\,.\; \textit{Th. VI}, (1) \div (2).$$

$$\therefore \quad \frac{\tan \tfrac{1}{2} A + \tan \tfrac{1}{2} B}{\tan \tfrac{1}{2} A - \tan \tfrac{1}{2} B} = \frac{\sin (\tfrac{1}{2} s - b) + \sin (\tfrac{1}{2} s - a)}{\sin (\tfrac{1}{2} s - b) - \sin (\tfrac{1}{2} s - a)}.$$

But $$\frac{\tan\frac{1}{2}A+\tan\frac{1}{2}B}{\tan\frac{1}{2}A-\tan\frac{1}{2}B}=\frac{\frac{\sin\frac{1}{2}A}{\cos\frac{1}{2}A}+\frac{\sin\frac{1}{2}B}{\cos\frac{1}{2}B}}{\frac{\sin\frac{1}{2}A}{\cos\frac{1}{2}A}-\frac{\sin\frac{1}{2}B}{\cos\frac{1}{2}B}}$$

$$=\frac{\frac{\sin\frac{1}{2}A}{\cos\frac{1}{2}A}+\frac{\sin\frac{1}{2}B}{\cos\frac{1}{2}B}}{\frac{\sin\frac{1}{2}A}{\cos\frac{1}{2}A}-\frac{\sin\frac{1}{2}B}{\cos\frac{1}{2}B}}\times\frac{\cos\frac{1}{2}A\cos\frac{1}{2}B}{\cos\frac{1}{2}A\cos\frac{1}{2}B}$$

$$=\frac{\sin\frac{1}{2}A\cos\frac{1}{2}B+\cos\frac{1}{2}A\sin\frac{1}{2}B}{\sin\frac{1}{2}A\cos\frac{1}{2}B-\cos\frac{1}{2}A\sin\frac{1}{2}B}$$

$$=\frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)}.$$

But

$$\frac{\sin(\frac{1}{2}s-b)+\sin(\frac{1}{2}s-a)}{\sin(\frac{1}{2}s-b)-\sin(\frac{1}{2}s-a)}=\frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a-b)}.$$

$$\therefore\quad \frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)}=\frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a-b)}.$$

$$\therefore \sin\tfrac{1}{2}(A+B) : \sin\tfrac{1}{2}(A-B) :: \tan\tfrac{1}{2}c : \tan\tfrac{1}{2}(a-b).$$

Cor. — Passing to the polar triangle, we have

$$\sin\tfrac{1}{2}(a+b) : \sin\tfrac{1}{2}(a-b) :: \cot\tfrac{1}{2}C : \tan\tfrac{1}{2}(A-B).$$

THEOREM XI.

In any spherical triangle, the cosine of one half the sum of any two angles is to the cosine of one half their difference as the tangent of one half the included side is to the tangent of one half the sum of the opposite sides.

For,

$$\frac{1}{\tan\frac{1}{2}A\tan\frac{1}{2}B}=\frac{\sin\frac{1}{2}s}{\sin(\frac{1}{2}s-c)}.\quad\left\{\begin{array}{r}\textit{Reciprocal of } (1)\times\\(2),\ \textit{Th. VI.}\end{array}\right.$$

$$\therefore\ \frac{1-\tan\frac{1}{2}A\tan\frac{1}{2}B}{1+\tan\frac{1}{2}A\tan\frac{1}{2}B}=\frac{\sin\frac{1}{2}s-\sin(\frac{1}{2}s-c)}{\sin\frac{1}{2}s+\sin(\frac{1}{2}s-c)}.$$

Reducing, as in Theorem X,

$$\frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)}=\frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a+b)}.$$

$$\therefore\ \cos\tfrac{1}{2}(A+B):\cos\tfrac{1}{2}(A-B)::\tan\tfrac{1}{2}c:\tan\tfrac{1}{2}(a+b).$$

Cor. — Passing to the polar,

$$\cos\tfrac{1}{2}(a+b):\cos\tfrac{1}{2}(a-b)::\cot\tfrac{1}{2}C:\tan\tfrac{1}{2}(A+B).$$

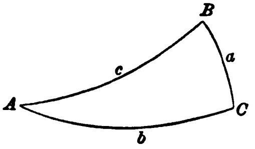
Scholium. — The results obtained in Theorems X and XI are known as *Napier's Analogies.*

THEOREM XII.

In any right spherical triangle, the sine of the middle part is equal to the product of the tangents of the adjacent parts.

For,

$$\cos A=\frac{\sin b\cot c-\sin A\cot C}{\cos b}\quad\ldots\quad\textit{Th. II, Cor. III.}$$



But $C=90°$.

$\therefore\ \cot C=0.$

$$\therefore\ \cos A=\frac{\sin b\cot c}{\cos b}=\tan b\cot c.$$

But $\cos A=\sin(90°-A)$,

and $\cot c=\tan(90°-c)$.

$$\therefore\ (1)\ \sin(90°-A)=\tan b\tan(90°-c).$$

Similarly,

$$\cos B = \tan a \cot c, \text{ and therefore}$$

$$(2)\ \sin(90° - B) = \tan a \tan(90° - c).$$

Again,

$$\cot A = \frac{\sin b \cot a - \cos b \cos C}{\sin C}.$$

But

$$C = 90°.$$

$$\therefore \quad \cot A = \sin b \cot a = \frac{\sin b}{\tan a},$$

$$\therefore \tan(90° - A) = \frac{\sin b}{\tan a}.$$

$$\therefore \quad (3)\ \sin b = \tan a \tan(90° - A).$$

Similarly,

$$(4)\ \sin a = \tan b \tan(90° - B).$$

Finally,

$$\cos c = \frac{\cos C + \cos A \cos B}{\sin A \sin B}. \quad \textit{Th. III}, (3).$$

But

$$C = 90°,$$

and

$$\cos c = \sin(90° - c).$$

$$\therefore \quad (5)\ \sin(90° - c) = \frac{\cos A \cos B}{\sin A \sin B} = \cot A \cot B$$
$$= \tan(90° - A) \tan(90° - B).$$

THEOREM XIII.

In any right spherical triangle, the sine of the middle part is equal to the product of the cosines of the opposite parts.

For,

$$\sin A = \frac{\cos B + \cos C \cos A}{\cos b \sin C} \quad . \quad . \quad \textit{Th. III}, (2).$$

But $C = 90°;$

$$\therefore \quad \sin A = \frac{\cos B}{\cos b}.$$

But $\sin A = \cos (90° - A),$

and $\cos B = \sin (90° - B).$

$\therefore$ (1) $\sin (90° - B) = \cos b \cos (90° - A).$

So, (2) $\sin (90° - A) = \cos a \cos (90° - B).$

Again,

$$\sin A = \frac{\sin a}{\sin c},$$

and $$\sin B = \frac{\sin b}{\sin c} \quad . \quad . \quad . \quad . \quad \textit{Th. I, Cor. II.}$$

Whence, $\sin a = \sin A \sin c,$

and $\sin b = \sin B \sin c.$

But $\sin A = \cos (90° - A),$

$\sin B = \cos (90° - B),$

and $\sin c = \cos (90° - c).$

$\therefore$ (3) $\sin a = \cos (90° - A) \cos (90° - c),$

and (4) $\sin b = \cos (90° - B) \cos (90° - c).$

Finally,

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

But $C = 90°,$

and $\cos c = \sin (90° - c).$

$\therefore$ (5) $\sin (90° - c) = \cos a \cos b.$

Scholium 1.—Theorems XI and XII are called *Napier's Principles.* They render it unnecessary to divide the subject of right spherical triangles into cases. In applying either, take the two given parts, if circular, otherwise their complements; and any one part required, if circular, otherwise its complement: observe which is the middle part, and whether the other parts are adjacent to, or opposite.

Scholium 2. — The formulas that demonstrate Napier's principles, give the following table, in which h stands for c, p for a, P for A. They can be committed and applied instead of the two rules. There will be no difficulty in remembering them, if they are associated with the corresponding formulas for plane triangles.

PLANE.

$$\sin P = \frac{p}{h}, \qquad \sin B = \frac{b}{h}.$$

$$\cos P = \frac{b}{h}, \qquad \cos B = \frac{p}{h}.$$

$$\tan P = \frac{p}{b}, \qquad \tan B = \frac{b}{p}.$$

$$\sin P = \cos B, \qquad \sin B = \cos P.$$

$$h^2 = b^2 + p^2,$$

$$1 = \cot B \cot P.$$

SPHERICAL.

$$\sin P = \frac{\sin p}{\sin h}, \qquad \sin B = \frac{\sin b}{\sin h}.$$

$$\cos P = \frac{\tan b}{\tan h}, \qquad \cos B = \frac{\tan p}{\tan h}.$$

$$\tan P = \frac{\tan p}{\sin b}, \qquad \tan B = \frac{\tan b}{\sin p}.$$

$$\sin P = \frac{\cos B}{\cos b}, \qquad \sin B = \frac{\cos P}{\cos p}.$$

$$\cos h = \cos b \cos p,$$

$$\cos h = \cot B \cot P.$$

Scholium 3.—In the application of the Napier rules, it will aid the memory to observe the common vowels in the first syllables of the words,—'*tan*gent' and '*ad*jacent,' '*co*sine' and '*op*posite.'

Scholium 4.—If the part sought is found in terms of its cosine, tangent, or cotangent, its species can be determined by the algebraic signs of the functions in the formula used; for when one of these functions is +, the corresponding angle is $< 90°$ (if $< 180°$), and when −, the angle is $> 90°$.

Scholium 5.—In applying Napier's rules for circular parts, however, the part sought will be determined by its sine. But the same sine corresponds to two different angles or arcs, supplements of each other. It becomes necessary, therefore, to discover such relations between the given and the required parts as will indicate which of the two arcs is to be taken.

THEOREM XIV.

In a right spherical triangle, the sides adjacent to the right angle are of the same species as their opposite angles.

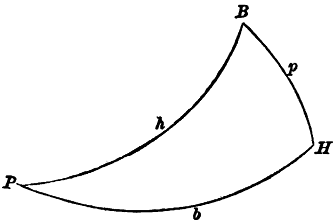
For,

$$\sin P = \frac{\cos B}{\cos b},$$

and $$\sin B = \frac{\cos P}{\cos p}.$$

But $$P < 180°,$$

and $$B < 180°.$$



Hence, $\sin P$ is $+$, and $\sin B$ is $+$.

Hence $\cos B$ and $\cos b$ must have like signs, so also $\cos P$ and $\cos p$. But if the functions have like signs, the corresponding quantities are of the same species. That is, B and b are both less, or both greater, than 90°; so also are P and p.

THEOREM XV.

If, in a right spherical triangle, the hypothenuse is less than 90°, *the two sides adjacent to the right angle are of the same species; so also are their opposite angles.*

For,

$$\cos h = \cos b \cos p.$$

If $h < 90°$, $\cos h$ is positive.

$\therefore$ $\cos b$ and $\cos p$ have like signs.

$\therefore$ b and p are of the same species.

$\therefore$ B and p are of the same species . . *Th. XIV.*

THEOREM XVI.

If, in a right spherical triangle, the hypothenuse is greater than 90°, *the two sides adjacent to the right angle are of different species; so also are their opposite angles.*

For,

$$\cos h = \cos b \cos p.$$

If $h > 90°$, $\cos h$ is negative,

and $\cos b \cos p$ is negative.

$\therefore$ $\cos b$ and $\cos p$ have unlike signs.

$\therefore$ b and p are of different species.

$\therefore$ B and P are of different species.

Scholium. — Theorems XIV, XV, XVI, enable us to determine, in the solution of right triangles, the nature of the part sought, except when an oblique angle and the opposite side are given to find the remaining parts.

THEOREM XVII.

When in a right spherical triangle a side (p) adjacent to the right angle and its opposite angle (P) are given, then:

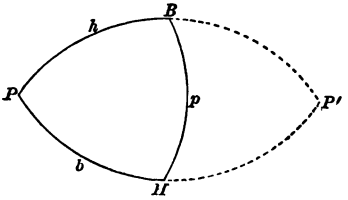
1. *If p differs more from 90° than P, there will be two solutions.*

2. *If $p = P$, and $P < 90°$ or $> 90°$, there will be one solution.*

3. *If $p = P = 90°$, there will be an indefinite number of solutions.*

4. *If p differs less from 90° than P, there will be no solution.*

1. Suppose $P < 90°$ and $p < P$. Produce PB and PH to meet in P'. PBP' and PHP' are semicircumferences, and $P = P'$. The triangles PHB, $P'HB$, both contain the given parts, and $P'B$, $P'H$, are each supplementary.



$$\sin b = \tan p \tan (90° - P)$$
$$= \tan p \cot P \text{ . } \textit{Th. XII.}$$

$$\tan p < \tan P, \text{ since } P < 90° \text{ and } p < P.$$

But, $\tan P \cot P = 1$ . . . .

$\therefore$ $\tan p \cot P < 1$.

$\therefore$ $\sin b < 1$.

$b < 90°$ or $> 90°$. Hence

b may be either $P\,H$ or $P'\,H$.

If $b < 90°$, $h < 90°$, since $p < 90°$. *Th. XV.*

If $b > 90°$, $h > 90°$, since $p < 90°$. *Th. XVI.*

Let $P > 90°$ and $p > P$.

$\sin b = \tan p \cot P$, as before,

$\tan p$, $\cot P$, are both negative,

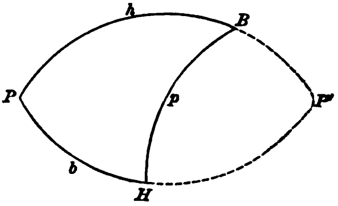
and

$\tan p < \tan P$, numerically.

But, $\tan P \cot P = 1$.

$\therefore$ $\tan p \cot P < 1$.

$\therefore$ $\sin b < 1$.



$\therefore$ $b < 90°$ or $> 90°$. Hence b may be either $P\,H$ or $P'\,H$.

If $b < 90°$, $h > 90°$, since $p > 90°$ . . . . . *Th. XVI.*

If $b > 90°$, $h < 90°$, since $p > 90°$ . . . . . *Th. XV.*

2. Suppose $P < 90°$ and $p = P$.

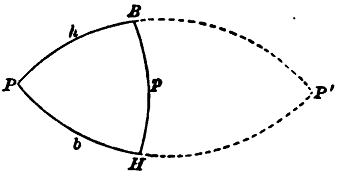
$\sin b = \tan p \cot P$.

But $p = P$.

$\therefore$ $\tan p \cot P = 1$.

$\therefore$ $\sin b = 1$.

$\therefore$ $b = 90°$.



$\sin\,(90° - h) = \cos b \cos p$,

or $\cos h = \cos b \cos p$.

But $$b = 90^\circ.$$

$$\therefore \quad \cos h = 0.$$

$$\therefore \quad h = 90^\circ.$$

$$\therefore \quad B = H = 90^\circ, \text{ by geometry.}$$

Hence, although there are two triangles, each is bi-rectangular, and equal in all its parts to the other.

Let $$P > 90^\circ \quad \text{and} \quad p = P.$$

$$\sin b = \tan p \cot P = \tan P \cot P = 1.$$

$$\therefore \quad b = 90^\circ.$$

$$\therefore \quad \cos h = \cos b \cos p = 0.$$

$$\therefore \quad h = 90^\circ.$$

$$\therefore \quad B = 90^\circ, \text{ the same result as before.}$$

3. Suppose $$P = 90^\circ.$$

$$\cos h = \cot B \cot P \quad . \quad \textit{Th. XIII, Sch. III.}$$

But $$\cot P = 0.$$

$$\therefore \quad \cos h = 0.$$

$$\therefore \quad h = 90^\circ.$$

$$\therefore \quad \sin p = \sin P \sin h = 1.$$

$$p = 90^\circ.$$

$$\sin B = \frac{\sin b}{\sin h} = \sin b, \text{ since } h = 90^\circ.$$

$$\therefore \quad B = b.$$

But $$\sin B = \frac{\cos P}{\cos p} = \frac{0}{0}, \text{ indeterminate,}$$

and $$\sin b = \frac{\tan P}{\tan p} = \frac{\infty}{\infty}, \text{ indeterminate,}$$

That is, there is an indefinite number of triangles which satisfy the given values of P and p.

4. Suppose $P < 90^\circ$ and $p > P$.

$$\sin b = \tan p \cot P.$$

$p < 90^\circ$, since p and P are of the same species.

$\therefore \quad \tan p > \tan P$, if $p > P$.

$\therefore \quad \tan p \cot P > \tan P \cot P$.

But $\tan P \cot P = 1$.

$\therefore \quad \tan p \cot P > 1$.

$\therefore \quad \sin b > 1$, which is impossible.

Let $P > 90^\circ$ and $p < P$.

$$\sin b = \tan p \cot P.$$

$p > 90^\circ$, since p and P are of the same species.

$\therefore \quad \tan p$, $\cot P$, are both negative,

and $\tan p > \tan P$, numerically.

$\therefore \quad \tan p \cot P > 1$, since $\tan P \cot P = 1$.

$\therefore \quad \sin b > 1$, which is impossible.

Hence, under these conditions, no triangle will satisfy the given values.

Scholium 1.—The preceding discussion gives the following table:

Two solutions when $\begin{cases} P < 90^\circ \text{ and } p < P, \\ P > 90^\circ \text{ and } p > P. \end{cases}$

One solution when $\begin{cases} P < 90^\circ \text{ and } p = P. \\ P > 90^\circ \text{ and } p = P. \end{cases}$

Indeterminate when $P = 90^\circ$.

No solution when $\begin{cases} P < 90^\circ \text{ and } p > P, \\ P > 90^\circ \text{ and } p < P. \end{cases}$

Scholium 2.—It will be noticed that the fourth result (no solution) harmonizes with the truth of theorem XIV. In fact, the case is inadmissible, since by that theorem p and P must be of the same species.

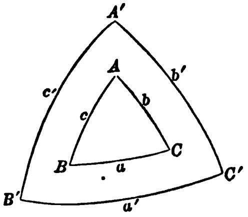
THEOREM XVIII.

Each angle of a spherical triangle is greater than the difference between 180° and the sum of the other angles.

For,

$A + B + C > 180°$, by geometry.

$$A > 180° - (B + C),$$
$$B > 180° - (A + C),$$
$$C > 180° - (A + B).$$



If $B + C > 180°$,

we have in the polar,

$$a' < b' + c' \text{ by geometry;}$$

or $$180° - A < 180° - B + 180° - C,$$

or $$-A < 180° - (B + C),$$

or $$A > B + C - 180°.$$

THEOREM XIX.

A side which differs more from 90° than another side, is of the same species as its opposite angle.

For,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} \quad . \quad . \quad . \quad . \quad . \quad . \quad Th.\ II,\ (1).$$

$b < 180°$ and $c < 180°$, hence $\sin b \sin c$ is positive.

If a differs more from 90° than b or c,

$$\cos a > \cos b \text{ or } > \cos c,$$

and still more, $\cos a > \cos b \cos c$, since $\cos b$, $\cos c$, are each less than 1.

Hence $\cos a$ will determine the sign of the fraction.

Hence $\cos A$ and $\cos a$ have the same sign.

Hence A and a are of the same species.

THEOREM XX.

An angle which differs more from 90° than another angle, is of the same species as its opposite side.

For

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C},$$

in which if A differs more from 90° than B or C, $\cos A$ determines the sign of the fraction, and therefore the sign of $\cos a$.

THEOREM XXI.

At least two sides are of the same species as their opposite angles.

This follows from Theorems XIX and XX.

THEOREM XXII.

The sum of two sides is greater than, equal to, or less than 180°, according as the sum of their opposite angles is greater than, equal to, or less than 180°.

For

$$\tan \tfrac{1}{2}(a+b) \cos \tfrac{1}{2}(A+B) = \tan \tfrac{1}{2} c \cos \tfrac{1}{2}(A-B) \text{ . } \textit{Th. XI.}$$

But the second member of this equation is always positive. Hence $\tan \frac{1}{2}(a+b)$ and $\cos \frac{1}{2}(A+B)$ must have like signs.

Hence, if

$$\tfrac{1}{2}(A+B) >, =, \text{ or } < 90^\circ,\ \tfrac{1}{2}(a+b) >, =, \text{ or } < 90^\circ.$$

Hence, if

$$A+B >, =, \text{ or } < 180^\circ,\ a+b >, =, \text{ or } < 180^\circ.$$

Scholium. — In the solution of oblique triangles, a required part may sometimes be found by its sine; in which case, unless the proper value can be determined by other considerations, there will be two values of that part answering to the conditions. In addition to Theorems XVIII–XXII, the following principles, established in geometry, will aid in selecting the true value.

1. *The greater side is opposite the greater angle, and conversely.*

2. *Each side is less than the sum of the other two.*

3. *The sum of the sides is less than 360°.*

4. *The sum of the angles is greater than 180°, and less than 540°.*

We are considering, it is understood, only those triangles, each of whose parts is less than 180°.

THEOREM XXIII.

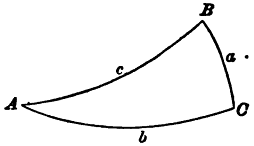
When, in an oblique spherical triangle, two sides and an angle opposite one of them are given, then:

1. *If the side opposite the given angle differs more from 90° than the other given side, there will be two solutions.*

2. *If the side opposite the given angle differs less from 90° than the other given side, there will be one solution.*

3. *If the side opposite the given angle differs more from 90° than the other given side, but is not of the same species as the given angle, there will be no solution.*

Let ABC be a spherical triangle, in which two sides, a and c, and an angle opposite one of them, A, are given.



$$1. \qquad \sin b = \frac{\cos a - \cos b \cos c}{\sin c \cos A} \quad . \quad . \quad Th.\ II,\ (1).$$

When a differs more from 90° than c,

$$\cos a > \cos c > \cos b \cos c;$$

and therefore $\cos a$ determines the sign of the numerator. But a and A are of the same species (*Th. XIX*). Hence $\cos a$ and $\cos A$ have like signs. Therefore both values of $\sin b$ are positive.

2. If a differs less from 90° than c (or if c differs more from 90° than a), C must be in the same quadrant as c, that is, of the same species; and since but one of

the two values of C can satisfy this condition, there will be but one solution. B and b will be found, in such case, to have but one admissible value.

3. If a differs more from 90° than c, but is not of the same species as A, $\cos a$ and $\cos A$ will have unlike signs.

Hence $\dfrac{\cos a - \cos b \cos c}{\sin c \cos A}$ will be negative;

$\therefore$ $\sin b$ will be negative, which is impossible, since the parts of the triangle are taken less than 180°.

Scholium. — The problem will be impossible also when the sine of the side opposite the given angle is less than the sine of the given angle into the sine of the given adjacent side.

For

$$\sin B = \frac{\sin A \sin b}{\sin a}.$$

But, if

$$\sin a < \sin A \sin b,$$

$$\sin B > 1, \text{ which is impossible.}$$

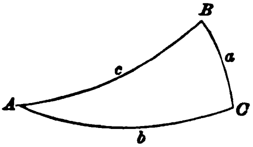
THEOREM XXIV.

When, in an oblique spherical triangle, two angles and a side opposite one of them are given, then:

1. *If the angle opposite the given side differs more from 90° than the other given angle, there will be two solutions.*

2. *If the angle opposite the given side differs less from 90° than the other given angle, there will be one solution.*

3. *If the angle opposite the given side differs more from 90° than the other given angle, and yet is in a different quadrant from the given side, there will be no solution.*



Let $A\,B\,C$ be a spherical triangle, in which two angles, A and B, and a side opposite one of them, b, are given.

1. $$\sin C = \frac{\cos B + \cos A \cos C}{\sin A \cos b} \quad . \quad . \quad \textit{Th. III}, (2).$$

When B differs more from 90° than A,

$$\cos B > \cos A > \cos A \cos C;$$

and therefore $\cos B$ determines the sign of the numerator. But B and b are of the same species (*Th. XX*). Hence $\cos B$ and $\cos b$ have like signs. Consequently the value of the fraction, or of $\sin C$, is always positive.

2. When B differs less from 90° than A (or when A differs more from 90° than B), a must be of the same species as A. But, since the two values of a are supplements, only one of them can satisfy this condition. C and c will be found, in such case, to have but one admissible value.

3. When B differs more from 90° than A, but is not of the same species as b, $\cos B$ and $\cos b$ will have unlike signs. Hence

$$\frac{\cos B + \cos A \cos C}{\sin A \cos b} \text{ will be negative;}$$

$\therefore$ sin C will be negative, which is impossible, since the parts are assumed to be less than 180°.

Scholium. — The problem will be impossible also when the sine of the angle opposite the given side is less than the sine of the other given angle into the sine of the given side.

For
$$\sin a = \frac{\sin b \sin A}{\sin B}.$$

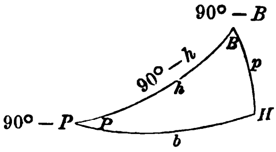
But, if
$$\sin B < \sin A \sin b,$$
$$\sin a > 1, \text{ which is impossible.}$$

§ 4. RIGHT TRIANGLES.

ILLUSTRATIONS.

I. *Given the two sides adjacent to the right angle.*

Suppose $\begin{cases} b = 63° \ 15', \\ p = 55° \ 28'. \end{cases}$



$$b < 90°,$$
and
$$p < 90°.$$
$$\therefore \quad B < 90°,$$
and
$$P < 90°.$$
Hence
$$h < 90°.$$

$$(1) \quad \sin(90° - h) = \cos b \cos p.$$
$$\text{L.} \sin(90° - h) = \text{L.} \cos b + \text{L.} \cos p - 10.$$
$$\therefore \quad h = 75° \ 13' \ 1''.$$

II. *Given the hypothenuse and a side adjacent to the right angle.*

$$\text{Suppose} \begin{cases} h = 105^\circ\ 17'\ 29'', \\ p = 38^\circ\ 47'\ 11''. \end{cases}$$

$$\begin{aligned}
& & p &< 90^\circ. \\
& \therefore & P &< 90^\circ, \\
\text{and} & & h &> 90^\circ. \\
& \therefore & b &> 90^\circ, \text{ since } p < 90^\circ. \\
& \therefore & P &< 90^\circ, \\
\text{and} & & B &> 90^\circ.
\end{aligned}$$

$$\begin{aligned}
(1)\quad \text{L. } \cos B &= \text{L. } \tan p + \text{L. } \cot h - 10. \\
\therefore\quad B &= 102^\circ\ 41'\ 33''. \\
(2)\quad \text{L. } \sin b &= \text{L. } \cos (90^\circ - h) + \text{L. } \cos (90^\circ - B) - 10 \\
&= \text{L. } \sin h + \text{L. } \sin B - 10. \\
\therefore\quad b &= 109^\circ\ 46'\ 32''. \\
(3)\quad \text{L. } \cos P &= \text{L. } \sin C + \text{L. } \cos b - 10. \\
\therefore\quad P &= 40^\circ\ 29'\ 50''.
\end{aligned}$$

III. *Given an oblique angle and the opposite side.*

$$\text{Suppose} \begin{cases} P = 75^\circ\ 30', \\ p = 50^\circ\ 15'. \end{cases}$$

$$\begin{aligned}
& & P &< 90^\circ, \\
\text{and} & & p &< P. \qquad \therefore \text{ two solutions.}
\end{aligned}$$

$$\begin{aligned}
\sin b &= \tan p \tan (90^\circ - P) \\
&= \tan p \cot P. \\
(1)\quad \text{L. } \sin b &= \text{L. } \tan p + \text{L. } \cot P - 10. \\
\therefore\quad b &= 18^\circ\ 7'\ 2'', \text{ or } 161^\circ\ 52'\ 58''.
\end{aligned}$$

(2) $\text{L. } \cos h = \text{L. } \cos p + \text{L. } \cos b - 10.$

$\therefore\ h = 52^\circ\ 34'\ 31''$, or $127^\circ\ 25'\ 29''$.

(3) $\text{L. } \cos B = \text{L. } \tan p + \text{L. } \cot h - 10.$

$\therefore\ B = 23^\circ\ 3'\ 6''$, or $156^\circ\ 56'\ 54''$.

In a similar manner, all other cases may be solved.

§ 5. EXERCISES.

1. Given $\left\{\begin{array}{l} a = 36^\circ\ 27' \\ b = 43^\circ\ 32'\ 31'' \end{array}\right\}$ Required $\left\{\begin{array}{l} c \\ A \\ B \end{array}\right.$

2. Given $\left\{\begin{array}{l} h = 63^\circ\ 56' \\ p = 40^\circ \end{array}\right\}$ Required $\left\{\begin{array}{l} b \\ B \\ P \end{array}\right.$

3. Given $\left\{\begin{array}{l} B = 91^\circ\ 1'\ 47'' \\ b = 115^\circ\ 20' \end{array}\right\}$ Required $\left\{\begin{array}{l} h \\ p \\ P \end{array}\right.$

4. Given $\left\{\begin{array}{l} P = 80^\circ\ 10'\ 30'' \\ b = 155^\circ\ 46'\ 42''.7 \end{array}\right\}$ Required $\left\{\begin{array}{l} B \\ p \\ h \end{array}\right.$

5. Given $\left\{\begin{array}{l} P = 63^\circ\ 15'\ 12'' \\ B = 135^\circ\ 33'\ 39'' \end{array}\right\}$ Required $\left\{\begin{array}{l} p \\ b \\ h \end{array}\right.$

§ 6. OBLIQUE TRIANGLES.

ILLUSTRATIONS.

I. *Given two sides and the included angle.*

Suppose
$$\begin{cases} a = 88^\circ\ 12'\ 20'', \\ b = 124^\circ\ 7'\ 17'', \\ C = 50^\circ\ 2'\ 1''. \end{cases}$$

$$\tan \tfrac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b) \cot \frac{1}{2} C}{\cos \frac{1}{2}(a + b)} \quad . \quad . \quad \textit{Th. XI, Cor.}$$

$$\therefore \quad \tfrac{1}{2}(A + B) = 97^\circ \ 46' \ 35''.5.$$

$$\tan \tfrac{1}{2}(B - A) = \frac{\sin \frac{1}{2}(b - a) \cot \frac{1}{2} C}{\sin \frac{1}{2}(b + a)} \quad . \quad \left\{ \begin{array}{l} \textit{Th. X, Cor., Th.} \\ \textit{XXII, Sch. 1.} \end{array} \right.$$

$$\therefore \quad \tfrac{1}{2}(B - A) = 34^\circ \ 31' \ 23''.5.$$

$$\therefore \quad A = 63^\circ \ 15' \ 12'',$$

and $B = 132^\circ \ 17' \ 59''.$

$$\tan \tfrac{1}{2} c = \frac{\sin \frac{1}{2}(B + A) \tan \frac{1}{2}(b - a)}{\sin \frac{1}{2}(B - A)} \quad . \quad . \quad \textit{Th. X.}$$

$$\therefore \quad \tfrac{1}{2} c = 29^\circ \ 32' \ 12''.5.$$

$$\therefore \quad c = 59^\circ \ 4' \ 25''.$$

II. *Given two angles and the included side.*

Suppose $\left\{ \begin{array}{l} A = 81^\circ \ 38' \ 20'', \\ B = 70^\circ \ \ 9' \ 38'', \\ c \ = 59^\circ \ 16' \ 22''. \end{array} \right.$

$$\tan \tfrac{1}{2}(a + b) = \frac{\cos \frac{1}{2}(A - B) \tan \frac{1}{2} c}{\cos \frac{1}{2}(A + B)}.$$

$$\tan \tfrac{1}{2}(a - b) = \frac{\sin \frac{1}{2}(A - B) \tan \frac{1}{2} c}{\sin \frac{1}{2}(A + B)}.$$

$$\cot \tfrac{1}{2} C = \frac{\sin \frac{1}{2}(a + b) \tan \frac{1}{2}(A - B)}{\sin \frac{1}{2}(a - b)}.$$

$$\therefore \quad \left\{ \begin{array}{l} a = 70^\circ \ \ 4' \ 17'', \\ b = 63^\circ \ 21' \ 27'', \\ C = 64^\circ \ 46' \ 24''. \end{array} \right.$$

III. *Given the three sides.*

Suppose $\begin{cases} a = 60^\circ, \\ b = 137^\circ\ 20', \\ c = 116^\circ. \end{cases}$

If either side equals or exceeds the sum of the other two, or if the sum of the three sides equals or exceeds 360°, there is no triangle; otherwise there is one triangle, and but one.

If only one angle is required, the formulas of Theorem V will be found to involve the least work. If all the angles are to be computed, the tangent formulas are most convenient, as they use only four different logarithms.

$$\tan \tfrac{1}{2} A = \sqrt{\frac{\sin (\frac{1}{2} s - b) \sin (\frac{1}{2} s - c)}{\sin \frac{1}{2} s \sin (\frac{1}{2} s - a)}},$$

$$\tan \tfrac{1}{2} B = \sqrt{\frac{\sin (\frac{1}{2} s - a) \sin (\frac{1}{2} s - c)}{\sin \frac{1}{2} s \sin (\frac{1}{2} s - b)}},$$

$$\tan \tfrac{1}{2} C = \sqrt{\frac{\sin (\frac{1}{2} s - a) \sin (\frac{1}{2} s - b)}{\sin \frac{1}{2} s \sin (\frac{1}{2} s - c)}}.$$

Whence, by the application of logarithms,

$A = 73^\circ\ 2'.4$, $B = 131^\circ\ 32'.2$, and $C = 96^\circ\ 55'.4$.

IV. *Given the three angles.*

Suppose $\begin{cases} A = 20^\circ\ 9'\ 56'', \\ B = 55^\circ\ 52'\ 32'', \\ C = 114^\circ\ 20'\ 14''. \end{cases}$

If either angle does not exceed the difference between 180° and the sum of the other two, or if the sum of the angles does not lie between 180° and 540°, there is no triangle; otherwise there is one triangle, and but one.

As between tangent and cosine formulas, the same choice may be made as in the preceding case.

$$\tan \tfrac{1}{2} a = \sqrt{\frac{-\cos \frac{1}{2} S \cos (\frac{1}{2} S - A)}{\cos (\frac{1}{2} S - B) \cos (\frac{1}{2} S - C)}},$$

$$\tan \tfrac{1}{2} b = \sqrt{\frac{-\cos \frac{1}{2} S \cos (\frac{1}{2} S - B)}{\cos (\frac{1}{2} S - A) \cos (\frac{1}{2} S - C)}},$$

$$\tan \tfrac{1}{2} c = \sqrt{\frac{-\cos \frac{1}{2} S \cos (\frac{1}{2} S - C)}{\cos (\frac{1}{2} S - A) \cos (\frac{1}{2} S - B)}}.$$

Applying logarithms, we find

$$a = 20° \; 16' \; 38'',$$
$$b = 56° \; 19' \; 41'',$$
$$c = 66° \; 20' \; 43''.$$

V. *Given two sides and an angle opposite one of them.*

$$\text{Suppose } \begin{cases} a = 57° \; 36', \\ b = 31° \; 12', \\ A = 104° \; 25' \; 30''. \end{cases}$$

$90° - a < 90° - b$, hence but one solution.

$$B < 90° \quad \ldots\ldots\ldots\ldots \quad \textit{Th. XIX.}$$

$$\sin B = \frac{\sin A \sin b}{\sin a} \quad \ldots\ldots\ldots \quad \textit{Th. I.}$$

$$\text{L. } \sin B = \text{L. } \sin A + \text{L. } \sin b + \text{a. c. L. } \sin a - 10.$$

$$\therefore \; B = 36° \; 26' \; 23''.$$

$$\cot \tfrac{1}{2} C = \frac{\sin \tfrac{1}{2} (a + b) \tan (A - B)}{\sin \tfrac{1}{2} (a - b)}.$$

$$\therefore \quad C = 51^\circ\ 41'\ 44''.$$

$$\sin c = \frac{\sin a \sin C}{\sin A} \quad \ldots\ldots\ldots \quad \textit{Th. I.}$$

$$c < 90^\circ \quad \ldots\ldots\ldots\ldots \quad \textit{Th. XX.}$$

$$\therefore \quad c = 43^\circ\ 11'\ 16''.$$

$$\text{Suppose} \begin{cases} a = 60^\circ\ 20', \\ b = 80^\circ\ 35', \\ A = 38^\circ\ 25'. \end{cases}$$

$$90^\circ - a > 90^\circ - b, \quad \therefore \text{ two solutions} \quad . \quad \textit{Th. XIII.}$$

$$\sin B = \frac{\sin A \sin b}{\sin a}.$$

$$\therefore \quad B = 44^\circ\ 52'\ 5'', \qquad \text{or } 135^\circ\ 7'\ 55''.$$

$$\cot \tfrac{1}{2} C = \frac{\sin \tfrac{1}{2} (b + a) \tan \tfrac{1}{2} (B - A)}{\sin \tfrac{1}{2} (b - a)}.$$

$$C > 180^\circ - (A + B) \quad \ldots \quad \textit{Th. XVIII.}$$

$$\therefore \quad C = 146^\circ\ 22'\ 43'', \qquad \text{or } 18^\circ\ 50'\ 7''.$$

$$\sin c = \frac{\sin a \sin C}{\sin A}.$$

$$C - 90^\circ > 90^\circ - A.$$

$$\therefore \quad c = 129^\circ\ 15'\ 35'', \qquad \text{or } 26^\circ\ 50'\ 13''.$$

VI. *Given two angles and the side opposite one of them.*

$$\text{Suppose} \qquad \begin{cases} A = 50^\circ\ 12', \\ B = 58^\circ\ \ 8', \\ a = 62^\circ\ 42'. \end{cases}$$

$$90^\circ - A > 90^\circ - B, \quad \therefore \text{ two solutions . } \textit{Th. XXIV.}$$

$$\sin b = \frac{\sin a \sin B}{\sin A}.$$

$$\therefore \quad b = 79^\circ\ 12'\ 10'', \qquad \text{or } 100^\circ\ 47'\ 50''.$$

$$\tan \tfrac{1}{2} c = \frac{\sin \frac{1}{2}(B + A) \tan \frac{1}{2}(b - a)}{\sin \frac{1}{2}(B - A)}.$$

$$\therefore \quad c = 119^\circ\ 3'\ 26'', \qquad \text{or } 152^\circ\ 14'\ 18''.$$

$$\sin C = \frac{\sin B \sin c}{\sin b}.$$

$$\therefore \quad C = 130^\circ\ 54'\ 28'', \qquad \text{or } 156^\circ\ 15'\ 6''.$$

The species of parts are determined by aid of the principles in Theorems XVIII–XXII.

§ 7. EXERCISES.

1. Given $\left\{\begin{array}{l} a = 97^\circ\ 30'\ 20'' \\ b = 55^\circ\ 12'\ 10'' \\ C = 39^\circ\ 58' \end{array}\right\}$ Required $\left\{\begin{array}{l} c \\ A \\ B \end{array}\right.$

2. Given $\left\{\begin{array}{l} B = 153^\circ\ 17'\ 6'' \\ C = 87^\circ\ 43'\ 36'' \\ a = 86^\circ\ 15'\ 15'' \end{array}\right\}$ Required $\left\{\begin{array}{l} a \\ c \\ B \end{array}\right.$

3. Given $\left\{\begin{array}{l} a = 50^\circ\ 12'\ 4'' \\ b = 116^\circ\ 44'\ 48'' \\ c = 129^\circ\ 11'\ 42'' \end{array}\right\}$ Required $\left\{\begin{array}{l} A \\ B \\ C \end{array}\right.$

4. Given $\left\{\begin{array}{l} A = 4^\circ\ 23'\ 35'' \\ B = 8^\circ\ 28'\ 20'' \\ C = 172^\circ\ 17'\ 56'' \end{array}\right\}$ Required $\left\{\begin{array}{l} a \\ b \\ c \end{array}\right.$

5. Given $\left\{\begin{array}{l} a = 30^\circ\ 52'\ 36''.6 \\ b = 31^\circ\ 9'\ 16'' \\ A = 87^\circ\ 34'\ 12'' \end{array}\right\}$ Required $\left\{\begin{array}{l} A \\ B \\ c \end{array}\right.$

6. Given $\left\{\begin{array}{l} a = 43^\circ\ 27'\ 36'' \\ b = 82^\circ\ 58'\ 17'' \\ A = 29^\circ\ 32'\ 29'' \end{array}\right\}$ Required $\left\{\begin{array}{l} B \\ C \\ c \end{array}\right.$

7. Given $\left\{\begin{array}{l} A = 95^\circ\ 16' \\ B = 80^\circ\ 42'\ 10'' \\ a = 57^\circ\ 38' \end{array}\right\}$ Required $\left\{\begin{array}{l} b \\ c \\ C \end{array}\right.$

8. Given $\left\{\begin{array}{l} a = 120^\circ \\ b = 70^\circ \\ A = 130^\circ \end{array}\right\}$ Required $\left\{\begin{array}{l} B \\ C \\ c \end{array}\right.$

9. Given $\left\{\begin{array}{l} A = 61^\circ\ 37'\ 52''.7 \\ B = 139^\circ\ 54'\ 34''.4 \\ b = 150^\circ\ 17'\ 26''.2 \end{array}\right\}$ Required $\left\{\begin{array}{l} a \\ c \\ C \end{array}\right.$

TABLE I.

LOGARITHMS OF NUMBERS.

TABLE I.

N.	0	1	2	3	4	5	6	7	8	9	d.
0	—	0000	3010	4771	6021	6990	7782	8451	9031	9542	
1	0000	0414	0792	1139	1461	1761	2041	2304	2553	2788	
2	3010	3222	3424	3617	3802	3979	4150	4314	4472	4624	
3	4771	4914	5051	5185	5315	5441	5563	5682	5798	5911	
4	6021	6128	6232	6335	6435	6532	6628	6721	6812	6902	
5	6990	7076	7160	7243	7324	7404	7482	7559	7634	7709	
6	7782	7853	7924	7993	8062	8129	8195	8261	8325	8388	
7	8451	8513	8573	8633	8692	8751	8808	8865	8921	8976	
8	9031	9085	9138	9191	9243	9294	9345	9395	9445	9494	
9	9542	9590	9638	9685	9731	9777	9823	9868	9912	9956	
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	41
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	38
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	35
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	32
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	30
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	28
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	26
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	25
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	23
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	22
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	21
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	20
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	19
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	18
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	18
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	17
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	16
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	16
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	15
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	15
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	14
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	14
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	13
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	13
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	13
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	12
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	12
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	12
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	11
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	11
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	11
N.	0	1	2	3	4	5	6	7	8	9	d.

N.	0	1	2	3	4	5	6	7	8	9	d.
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	11
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	10
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	10
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	10
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	10
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	10
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	9
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	9
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	9
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	8
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	8
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	8
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	8
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	7
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	7
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	7
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	7
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	7
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	7
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	7
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	6
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	6
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	6
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	6
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	6
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	6
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	6
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	5
N.	0	1	2	3	4	5	6	7	8	9	d.

TABLE I.

N.	0	1	2	3	4	5	6	7	8	9	d.
80	9031	9036	9042	9047	9053	9058	9063	7069	9074	9079	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	5
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	5
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	5
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	5
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	5
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	5
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	5
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	5
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	5
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	4
100	0000	0004	0009	0013	0017	0022	0026	0030	0035	0039	4
101	0043	0048	0052	0056	0060	0065	0069	0073	0077	0082	4
102	0086	0090	0095	0099	0103	0107	0111	0116	0120	0124	4
103	0128	0133	0137	0141	0145	0149	0154	0158	0162	0166	4
104	0170	0175	0179	0183	0187	0191	0195	0199	0204	0208	4
105	0212	0216	0220	0224	0228	0233	0237	0241	0245	0249	4
106	0253	0257	0261	0265	0269	0273	0278	0282	0286	0290	4
107	0294	0298	0302	0306	0310	0314	0318	0322	0326	0330	4
108	0334	0338	0342	0346	0350	0354	0358	0362	0366	0370	4
109	0374	0378	0382	0386	0390	0394	0398	0402	0406	0410	4
110	0414	0418	0422	0426	0430	0434	0438	0441	0445	0449	4
111	0453	0457	0461	0465	0469	0473	0477	0481	0484	0488	4
112	0492	0496	0500	0504	0508	0512	0515	0519	0523	0527	4
113	0531	0535	0538	0542	0546	0550	0554	0558	0561	0565	4
114	0569	0573	0577	0580	0584	0588	0592	0596	0599	0603	4
115	0607	0611	0615	0618	0622	0626	0630	0633	0637	0641	4
116	0645	0648	0652	0656	0660	0663	0667	0671	0674	0678	4
117	0682	0686	0689	0693	0697	0700	0704	0708	0711	0715	4
118	0719	0722	0726	0730	0734	0737	0741	0745	0748	0752	4
119	0755	0759	0763	0766	0770	0774	0777	0781	0785	0788	4
120	0792	0795	0799	0803	0806	0810	0813	0817	0821	0824	4
N.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**d.**

N.	0	1	2	3	4	5	6	7	8	9	d.
120	0792	0795	0799	0803	0806	0810	0813	0817	0821	0824	4
121	0828	0831	0835	0839	0842	0846	0849	0853	0856	0860	4
122	0864	0867	0871	0874	0878	0881	0885	0888	0892	0896	4
123	0899	0903	0906	0910	0913	0917	0920	0924	0927	0931	3
124	0934	0938	0941	0945	0948	0952	0955	0959	0962	0966	3
125	0969	0973	0976	0980	0983	0986	0990	0993	0997	1000	3
126	1004	1007	1011	1014	1017	1021	1024	1028	1031	1035	3
127	1038	1041	1045	1048	1052	1055	1059	1062	1065	1069	3
128	1072	1075	1079	1082	1086	1089	1092	1096	1099	1103	3
129	1106	1109	1113	1116	1119	1123	1126	1129	1133	1136	3
130	1139	1143	1146	1149	1153	1156	1159	1163	1166	1169	3
131	1173	1176	1179	1183	1186	1189	1193	1196	1199	1202	3
132	1206	1209	1212	1216	1219	1222	1225	1229	1232	1235	3
133	1239	1242	1245	1248	1252	1255	1258	1261	1265	1268	3
134	1271	1274	1278	1281	1284	1287	1290	1294	1297	1300	3
135	1303	1307	1310	1313	1316	1319	1323	1326	1329	1332	3
136	1335	1339	1342	1345	1348	1351	1355	1358	1361	1364	3
137	1367	1370	1374	1377	1380	1383	1386	1389	1392	1396	3
138	1399	1402	1405	1408	1411	1414	1418	1421	1424	1427	3
139	1430	1433	1436	1440	1443	1446	1449	1452	1455	1458	3
140	1461	1464	1467	1471	1474	1477	1480	1483	1486	1489	3
141	1492	1495	1498	1501	1504	1508	1511	1514	1517	1520	3
142	1523	1526	1529	1532	1535	1538	1541	1544	1547	1550	3
143	1553	1556	1559	1562	1565	1569	1572	1575	1578	1581	3
144	1584	1587	1590	1593	1596	1599	1602	1605	1608	1611	3
145	1614	1617	1620	1623	1626	1629	1632	1635	1638	1641	3
146	1644	1647	1649	1652	1655	1658	1661	1664	1667	1670	3
147	1673	1676	1679	1682	1685	1688	1691	1694	1697	1700	3
148	1703	1706	1708	1711	1714	1717	1720	1723	1726	1729	3
149	1732	1735	1738	1741	1744	1746	1749	1752	1755	1758	3
150	1761	1764	1767	1770	1772	1775	1778	1781	1784	1787	3
151	1790	1793	1796	1798	1801	1804	1807	1810	1813	1816	3
152	1818	1821	1824	1827	1830	1833	1836	1838	1841	1844	3
153	1847	1850	1853	1855	1858	1861	1864	1867	1870	1872	3
154	1875	1878	1881	1884	1886	1889	1892	1895	1898	1901	3
155	1903	1906	1909	1912	1915	1917	1920	1923	1926	1928	3
156	1931	1934	1937	1940	1942	1945	1948	1951	1953	1956	3
157	1959	1962	1965	1967	1970	1973	1976	1978	1981	1984	3
158	1987	1989	1992	1995	1998	2000	2003	2006	2009	2011	3
159	2014	2017	2019	2022	2025	2028	2030	2033	2036	2038	3
160	2041	2044	2047	2049	2052	2055	2057	2060	2063	2066	3
N.	0	1	2	3	4	5	6	7	8	9	d.

N.	0	1	2	3	4	5	6	7	8	9	d.
160	2041	2044	2047	2049	2052	2055	2057	2060	2063	2066	3
161	2068	2071	2074	2076	2079	2082	2084	2087	2090	2092	3
162	2095	2098	2101	2103	2106	2109	2111	2114	2117	2119	3
163	2122	2125	2127	2130	2133	2135	2138	2140	2143	2146	3
164	2148	2151	2154	2156	2159	2162	2164	2167	2170	2172	3
165	2175	2177	2180	2183	2185	2188	2191	2193	2196	2198	3
166	2201	2204	2206	2209	2212	2214	2217	2219	2222	2225	3
167	2227	2230	2232	2235	2238	2240	2243	2245	2248	2251	3
168	2253	2256	2258	2261	2263	2266	2269	2271	2274	2276	3
169	2279	2281	2284	2287	2289	2292	2294	2297	2299	2302	3
170	2304	2307	2310	2312	2315	2317	2320	2322	2325	2327	3
171	2330	2333	2335	2338	2340	2343	2345	2348	2350	2353	2
172	2355	2358	2360	2363	2365	2368	2370	2373	2375	2378	2
173	2380	2383	2385	2388	2390	2393	2395	2398	2400	2403	2
174	2405	2408	2410	2413	2415	2418	2420	2423	2425	2428	2
175	2430	2433	2435	2438	2440	2443	2445	2448	2450	2453	2
176	2455	2458	2460	2463	2465	2467	2470	2472	2475	2477	2
177	2480	2482	2485	2487	2490	2492	2494	2497	2499	2502	2
178	2504	2507	2509	2512	2514	2516	2519	2521	2524	2526	2
179	2529	2531	2533	2536	2538	2541	2543	2545	2548	2550	2
180	2553	2555	2558	2560	2562	2565	2567	2570	2572	2574	2
181	2577	2579	2582	2584	2586	2589	2591	2594	2596	2598	2
182	2601	2603	2605	2608	2610	2613	2615	2617	2620	2622	2
183	2625	2627	2629	2632	2634	2636	2639	2641	2643	2646	2
184	2648	2651	2653	2655	2658	2660	2662	2665	2667	2669	2
185	2672	2674	2676	2679	2681	2683	2686	2688	2690	2693	2
186	2695	2697	2700	2702	2704	2707	2709	2711	2714	2716	2
187	2718	2721	2723	2725	2728	2730	2732	2735	2737	2739	2
188	2742	2744	2746	2749	2751	2753	2755	2758	2760	2762	2
189	2765	2767	2769	2772	2774	2776	2778	2781	2783	2785	2
190	2788	2790	2792	2794	2797	2799	2801	2804	2806	2808	2
191	2810	2813	2815	2817	2819	2822	2824	2826	2828	2831	2
192	2833	2835	2838	2840	2842	2844	2847	2849	2851	2853	2
193	2856	2858	2860	2862	2865	2867	2869	2871	2874	2876	2
194	2878	2880	2882	2885	2887	2889	2891	2894	2896	2898	2
195	2900	2903	2905	2907	2909	2911	2914	2916	2918	2920	2
196	2923	2925	2927	2929	2931	2934	2936	2938	2940	2942	2
197	2945	2947	2949	2951	2953	2956	2958	2960	2962	2964	2
198	2967	2969	2971	2973	2575	2978	2980	2982	2984	2986	2
199	2989	2991	2993	2995	2997	2999	3002	3004	3006	3008	2
200	3010	3012	3015	3017	3019	3021	3023	3025	3028	3030	2
N.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**d.**

TABLE II.

LOGARITHMS OF TRIGONOMETRIC FUNCTIONS.

Log $R = 10$.

L. Sin. **0° — 10°.**

° ′	0′	1′	2′	3′	4′	5′	6′	7′	8′	9′	10′	d.	
0 0		6.4637	6.7648	6.9408	7.0658	7.1627	7.2419	7.3088	7.3668	7.4180	7.4637		50
10	7.4637	7.5051	7.5429	7.5777	.6099	.6398	.6678	.6942	.7190	.7425	.7648		40
20	.7648	.7859	.8061	.8255	.8439	.8617	.8787	.8951	.9109	.9261	.9408		30
30	.9408	.9551	.9689	.9822	.9952	8.0078	8.0200	8.0319	8.0435	8.0548	8.0658		20
40	8.0658	8.0765	8.0870	8.0972	8.1072	.1169	.1265	.1358	.1450	.1539	.1627		10
50	.1627	.1713	.1797	.1880	.1961	.2041	.2119	.2196	.2271	.2346	.2419		0 **89**
1 0	.2419	.2490	.2561	.2630	.2699	.2766	.2832	.2898	.2962	.3025	.3088		50
10	.3088	.3150	.3210	.3270	.3329	.3388	.3445	.3502	.3558	.3613	.3668		40
20	.3668	.3722	.3775	.3828	.3880	.3931	.3982	.4032	.4082	.4131	.4179		30
30	.4179	.4227	.4275	.4322	.4368	.4414	.4459	.4504	.4549	.4593	.4637		20
40	.4637	.4680	.4723	.4765	.4807	.4848	.4890	.4930	.4971	.5011	.5050		10
50	.5050	.5090	.5129	.5167	.5206	.5243	.5281	.5318	.5355	.5392	.5428		0 **88**
2 0	.5428	.5464	.5500	.5535	.5571	.5605	.5640	.5674	.5708	.5742	.5776	35	50
10	.5776	.5809	.5842	.5875	.5907	.5939	.5972	.6003	.6035	.6066	.6097	32	40
20	.6097	.6128	.6159	.6189	.6220	.6250	.6279	.6309	.6339	.6368	.6397	30	30
30	.6397	.6426	.6454	.6483	.6511	.6539	.6567	.6595	.6622	.6650	.6677	28	20
40	.6677	.6704	.6731	.6758	.6784	.6810	.6837	.6863	.6889	.6914	.6940	26	10
50	.6940	.6965	.6991	.7016	.7041	.7066	.7090	.7115	.7140	.7164	.7188	25	0 **87**
3 0	.7188	.7212	.7236	.7260	.7283	.7307	.7330	.7354	.7377	.7400	.7423	23	50
10	.7423	.7445	.7468	.7491	.7513	.7535	.7557	.7580	.7602	.7623	.7645	22	40
20	.7645	.7667	.7688	.7710	.7731	.7752	.7773	.7794	.7815	.7836	.7857	21	30
30	.7857	.7877	.7898	.7918	.7939	.7959	.7979	.7999	.8019	.8039	.8059	20	20
40	.8059	.8078	.8098	.8117	.8137	.8156	.8175	.8194	.8213	.8232	.8251	19	10
50	.8251	.8270	.8289	.8307	.8326	.8345	.8363	.8381	.8400	.8418	.8436	18	0 **86**
4 0	.8436	.8454	.8472	.8490	.8508	.8525	.8543	.8560	.8578	.8595	.8613	18	50
10	.8613	.8630	.8647	.8665	.8682	.8699	.8716	.8733	.8749	.8766	.8783	17	40
20	.8783	.8799	.8816	.8833	.8849	.8865	.8882	.8898	.8914	.8930	.8946	16	30
30	.8946	.8962	.8978	.8994	.9010	.9026	.9042	.9057	.9073	.9089	.9104	16	20
40	.9104	.9119	.9135	.9150	.9166	.9181	.9196	.9211	.9226	.9241	.9256	15	10
50	.9256	.9271	.9286	.9301	.9315	.9330	.9345	.9359	.9374	.9388	.9403	15	0 **85**
5 0	.9403	.9417	.9432	.9446	.9460	.9475	.9489	.9503	.9517	.9531	.9545	14	50
10	9545	.9559	.9573	.9587	9601	.9614	.9628	.9642	.9655	.9669	.9682	14	40
20	.9682	.9696	.9709	.9723	.9736	.9750	.9763	.9776	.9789	.9803	.9816	13	30
30	.9816	.9829	.9842	.9855	.9868	.9881	.9894	.9907	.9919	.9932	.9945	13	20
40	.9945	.9958	.9970	.9983	.9996	9.0008	9.0021	9.0033	9.0046	9.0058	9.0070	12	10
50	9.0070	9.0083	9.0095	9.0107	9.0120	.0132	.0144	.0156	.0168	.0180	.0192	12	0 **84**
6 0	.0192	.0204	.0216	.0228	.0240	.0252	.0264	.0276	.0287	.0299	.0311	12	50
10	.0311	.0323	.0334	.0346	.0357	.0369	.0380	.0392	.0403	.0415	.0426	11	40
20	.0426	.0438	.0449	.0460	.0472	.0483	.0494	.0505	.0516	.0527	.0539	11	30
30	.0539	.0550	.0561	.0572	.0583	.0594	.0605	.0616	.0626	.0637	.0648	11	20
40	.0648	.0659	.0670	.0680	.0691	.0702	.0712	.0723	.0734	.0744	.0755	11	10
50	.0755	.0765	.0776	.0786	.0797	.0807	.0818	.0828	.0838	.0849	.0859	10	0 **83**
7 0	.0859	.0869	.0879	.0890	.0900	.0910	.0920	.0930	.0940	.0951	.0961	10	50
10	.0961	.0971	.0981	.0991	.1001	.1011	.1020	.1030	.1040	.1030	.1060	10	40
20	.1060	.1070	.1080	.1089	.1099	.1109	.1118	.1128	.1138	.1147	.1157	10	30
30	.1157	.1167	.1176	.1186	.1195	.1205	.1214	.1224	.1233	.1242	.1252	9	20
40	.1252	.1261	.1271	.1280	.1289	.1299	.1308	.1317	.1326	.1336	.1345	9	10
50	.1345	.1354	.1363	.1372	.1381	.1390	.1399	.1409	.1418	.1427	.1436	9	0 **82**
8 0	.1436	.1445	.1453	.1462	.1471	.1480	.1489	.1498	.1507	.1516	.1525	9	50
10	.1525	.1533	.1542	.1551	.1560	.1568	.1577	.1586	.1594	.1603	.1612	9	40
20	.1612	.1620	.1629	.1637	.1646	.1655	.1663	.1672	.1680	.1689	.1697	8	30
30	.1697	.1705	.1714	.1722	.1731	.1739	.1747	.1756	.1764	.1772	.1781	8	20
40	.1781	.1789	.1797	.1806	.1814	.1822	.1830	.1838	.1847	.1855	.1863	8	10
50	.1863	.1871	.1879	.1887	.1895	.1903	.1911	.1919	.1927	.1935	.1943	8	0 **81**
9 0	.1943	.1951	.1959	.1967	.1975	.1983	.1991	.1999	.2007	.2015	.2022	8	50
10	.2022	.2030	.2038	.2046	.2054	.2061	.2069	.2077	.2085	.2092	.2100	7	40
20	.2100	.2108	.2115	.2123	.2131	.2138	.2146	.2153	.2161	.2169	.2176	8	30
30	.2176	.2184	.2191	.2199	.2206	.2214	.2221	.2229	.2236	.2243	.2251	8	20
40	.2251	.2258	.2266	.2273	.2280	.2288	.2295	.2303	.2310	.2317	.2324	7	10
50	.2324	.2332	.2339	.2346	.2353	.2361	.2368	.2375	.2382	.2390	.2397	7	0 **80**
	10′	9′	8′	7′	6′	5′	4′	3′	2′	1′	0′	d.	

80° — 90°. **L. Cos.**

L. Tan. **0° — 10°.**

° ′	0′	1′	2′	3′	4′	5′	6′	7′	8′	9′	10′	d.	
0 0	—	6.4637	6.7648	6.9408	7.0658	7.1627	7.2419	7.3088	7.3668	7.4180	7.4637		50
10	7.4637	7.5051	7.5429	7.5777	.6099	.6398	.6678	.6942	.7190	.7425	.7648		40
20	.7648	.7860	.8062	.8255	.8439	.8617	.8787	.8951	.9109	.9261	.9409		30
30	.9409	.9551	.9689	.9823	.9952	8.0078	8.0200	8.0319	8.0435	8.0548	8.0658		20
40	8.0658	8.0765	8.0870	8.0972	8.1072	.1170	.1265	.1359	.1450	.1540	.1627		10
50	.1627	.1713	.1798	.1880	.1962	.2041	.2120	.2196	.2272	.2346	.2419		0 **89**
1 0	.2419	.2491	.2562	.2631	.2700	.2767	.2833	.2899	.2963	.3026	.3089		50
10	.3089	.3150	.3211	.3271	.3330	.3389	.3446	.3503	.3559	.3614	.3669		40
20	.3669	.3723	.3776	.3829	.3881	.3932	.3983	.4033	.4083	.4132	.4181		30
30	.4181	.4229	.4276	.4323	.4370	.4416	.4461	.4506	.4551	.4595	.4638		20
40	.4638	.4682	.4725	.4767	.4809	.4851	.4892	.4933	.4973	.5013	.5053		10
50	.5053	.5092	.5131	.5170	.5208	.5246	.5283	.5321	.5358	.5394	.5431		0 **88**
2 0	.5431	.5467	.5503	.5538	.5573	.5608	.5643	.5677	.5711	.5745	.5779	35	50
10	.5779	.5812	.5845	.5878	.5911	.5943	.5975	.6007	.6038	.6070	.6101	32	40
20	.6101	.6132	.6163	.6193	.6223	.6254	.6283	.6313	.6343	.6372	.6401	30	30
30	.6401	.6430	.6459	.6487	.6515	.6544	.6571	.6599	.6627	.6654	.6682	28	20
40	.6682	.6709	.6736	.6762	.6789	.6815	.6842	.6868	.6894	.6920	.6945	26	10
50	.6945	.6971	.6996	.7021	.7046	.7071	.7096	.7121	.7145	.7170	.7194	25	0 **87**
3 0	.7194	.7218	.7242	.7266	.7290	.7313	.7337	.7360	.7383	.7406	.7429	23	50
10	.7429	.7452	.7475	.7497	.7520	.7542	.7565	.7587	.7609	.7631	.7652	22	40
20	.7652	.7674	.7696	.7717	.7739	.7760	.7781	.7802	.7823	.7844	.7865	21	30
30	.7865	.7885	.7906	.7927	.7947	.7967	.7988	.8008	.8028	.8048	.8067	20	20
40	.8067	.8087	.8107	.8126	.8146	.8165	.8185	.8204	.8223	.8242	.8261	19	10
50	.8261	.8280	.8299	.8317	.8336	.8355	.8373	.8392	.8410	.8428	.8446	18	0 **86**
4 0	.8446	.8465	.8483	.8501	.8518	.8536	.8554	.8572	.8589	.8607	.8624	18	50
10	.8624	.8642	.8659	.8676	.8694	.8711	.8728	.8745	.8762	.8778	.8795	17	40
20	.8795	.8812	.8829	.8845	.8862	.8878	.8895	.8911	.8927	.8944	.8960	16	30
30	.8960	.8976	.8992	.9008	.9024	.9040	.9056	.9071	.9087	.9103	.9118	16	20
40	.9118	.9134	.9150	.9165	.9180	.9196	.9211	.9226	.9241	.9256	.9272	15	10
50	.9272	.9287	.9302	.9316	.9331	.9346	.9361	.9376	.9390	.9405	.9420	15	0 **85**
5 0	.9420	.9434	.9449	.9463	.9477	.9492	.9506	.9520	.9534	.9549	.9563	14	50
10	.9563	.9577	.9591	.9605	.9619	.9633	.9646	.9660	.9674	.9688	.9701	14	40
20	.9701	.9715	.9729	.9742	.9756	.9769	.9782	.9796	.9809	.9823	.9836	13	30
30	.9836	.9849	.9862	.9875	.9888	.9901	.9915	.9928	.9940	.9953	.9966	13	20
40	.9966	.9979	.9992	9.0005	9.0017	9.0030	9.0043	9.0055	9.0068	9.0080	9.0093	13	10
50	9.0093	9.0105	9.0118	.0130	.0143	.0155	.0167	.0180	.0192	.0204	.0216	12	0 **84**
6 0	.0216	.0228	.0240	.0253	.0265	.0277	.0289	.0300	.0312	.0324	.0336	12	50
10	.0336	.0348	.0360	.0371	.0383	.0395	.0407	.0418	.0430	.0441	.0453	12	40
20	.0453	.0464	.0476	.0487	.0499	.0510	.0521	.0533	.0544	.0555	.0567	11	30
30	.0567	.0578	.0589	.0600	.0611	.0622	.0633	.0645	.0656	.0667	.0678	11	20
40	.0678	.0688	.0699	.0710	.0721	.0732	.0743	.0754	.0764	.0775	.0786	11	10
50	.0786	.0796	.0807	.0818	.0828	.0839	.0849	.0860	.0871	.0881	.0891	10	0 **83**
7 0	.0891	.0902	.0912	.0923	.0933	.0943	.0954	.0964	.0974	.0984	.0995	10	50
10	.0995	.1005	.1015	.1025	.1035	.1045	.1055	.1066	.1076	.1086	.1096	10	40
20	.1096	.1106	.1116	.1125	.1135	.1145	.1155	.1165	.1175	.1185	.1194	10	30
30	.1194	.1204	.1214	.1223	.1233	.1243	.1252	.1262	.1272	.1281	.1291	10	20
40	.1291	.1300	.1310	.1319	.1329	.1338	.1348	.1357	.1367	.1376	.1385	9	10
50	.1385	.1395	.1404	.1413	.1423	.1432	.1441	.1450	.1460	.1469	.1478	9	0 **82**
8 0	.1478	.1487	.1496	.1505	.1515	.1524	.1533	.1542	.1551	.1560	.1569	9	50
10	.1569	.1578	.1587	.1596	.1605	.1613	.1622	.1631	.1640	.1649	.1658	9	40
20	.1658	.1667	.1675	.1684	.1693	.1702	.1710	.1719	.1728	.1736	.1745	9	30
30	.1745	.1754	.1762	.1771	.1779	.1788	.1797	.1805	.1814	.1822	.1831	9	20
40	.1831	.1839	.1848	.1856	.1864	.1873	.1881	.1890	.1898	.1906	.1915	8	10
50	.1915	.1923	.1931	.1940	.1948	.1956	.1964	.1973	.1981	.1989	.1997	8	0 **81**
9 0	.1997	.2005	.2013	.2022	.2030	.2038	.2046	.2054	.2062	.2070	.2078	8	50
10	.2078	.2086	.2094	.2102	.2110	.2118	.2126	.2134	.2142	.2150	.2158	8	40
20	.2158	.2166	.2174	.2181	.2189	.2197	.2205	.2213	.2221	.2228	.2236	8	30
30	.2236	.2244	.2252	.2259	.2267	.2275	.2282	.2290	.2298	.2305	.2313	8	20
40	.2313	.2321	.2328	.2336	.2343	.2351	.2359	.2366	.2374	.2381	.2389	8	10
50	.2389	.2396	.2404	.2411	.2419	.2426	.2434	.2441	.2448	.2456	.2463	7	0 **80**
	10′	9′	8′	7′	6′	5′	4′	3′	2′	1′	0′	d.	

80° — 90°. **L. Cot.**

° ′	L. Sin.	d.	L. Tan.	d. c.	L. Cot.	L. Cos.	d.	
0 0	—		—		—	0.0000	0	0 **90**
10	7.4637	3011	7.4637	3011	12.5363	0.0000	0	50
20	7.7648	1760	7.7648	1761	12.2352	0.0000	0	40
30	7.9408	1250	7.9409	1249	12.0591	0 0000	0	30
40	8.0658	969	8.0658	969	11.9342	0.0000	0	20
50	8.1627	792	8.1627	792	11.8373	0.0000	1	10
1 0	8.2419	669	8.2419	670	11.7581	9.9999	0	0 **89**
10	8.3088	580	8.3089	580	11.6911	9.9999	0	50
20	8.3668	511	8.3669	512	11.6331	9.9999	0	40
30	8.4179	458	8.4181	457	11.5819	9.9999	1	30
40	8.4637	413	8.4638	415	11.5362	9.9998	0	20
50	8.5050	378	8.5053	378	11.4947	9.9998	1	10
2 0	8.5428	348	8.5431	348	11.4569	9.9997	0	0 **88**
10	8.5776	321	8.5779	322	11.4221	9.9997	1	50
20	8.6097	300	8.6101	300	11.3899	9.9996	0	40
30	8.6397	280	8.6401	281	11.3599	9.9996	1	30
40	8.6677	263	8.6682	263	11.3318	9.9995	0	20
50	8 6940	248	8.6945	249	11.3055	9.9995	1	10
3 0	8.7188	235	8.7194	235	11.2806	9.9994	1	0 **87**
10	8.7423	222	8.7429	223	11.2571	9.9993	0	50
20	8.7645	212	8 7652	213	11.2348	9.9993	1	40
30	8.7857	202	8.7865	202	11.2135	9.9992	1	30
40	8.8059	192	8.8067	194	11.1933	9.9991	1	20
50	8.8251	185	8.8261	185	11.1739	9.9990	1	10
4 0	8.8436	177	8.8446	178	11.1554	9.9989	0	0 **86**
10	8.8613	170	8.8624	171	11.1376	9.9989	1	50
20	8.8783	163	8.8795	165	11.1205	9.9988	1	40
30	8.8946	158	8.8960	158	11.1040	9.9987	1	30
40	8.9104	152	8.9118	154	11.0882	9.9986	1	20
50	8.9256	147	8.9272	148	11.0728	9.9985	2	10
5 0	8.9403	142	8.9420	143	11.0580	9.9983	1	0 **85**
10	8.9545	137	8.9563	138	11.0437	9.9982	1	50
20	8.9682	134	8.9701	135	11.0299	9.9981	1	40
30	8.9816	129	8.9836	130	11.0164	9.9980	1	30
40	8.9945	125	8.9966	127	11.0034	9.9979	2	20
50	9.0070	122	9.0093	123	10.9907	9.9977	1	10
6 0	9.0192	119	9.0216	120	10.9784	9.9976	1	0 **84**
10	9.0311	115	9.0336	117	10.9664	9.9975	2	50
20	9.0426	113	9.0453	114	10.9547	9.9973	1	40
30	9.0539	109	9.0567	111	10.9433	9.9972	1	30
40	9.0648	107	9.0678	108	10.9322	9.9971	2	20
50	9.0755	104	9.0786	105	10.9214	9.9969	1	10
7 0	9.0859	102	9.0891	104	10.9109	9.9968	2	0 **83**
10	9.0961	99	9.0995	101	10.9005	9.9966	2	50
20	9.1060	97	9.1096	98	10.8904	9.9964	1	40
30	9.1157	95	9.1194	97	10.8806	9.9963	2	30
40	9.1252	93	9.1291	94	10.8709	9.9961	2	20
50	9.1345	91	9.1385	93	10.8615	9.9959	1	10
8 0	9.1436		9.1478		10.8522	9.9958		0 **82**
	L. Cos.	d.	L. Cot.	d. c.	L. Tan.	L. Sin.	d.	′ °

° ′	L. Sin.	d.	L. Tan.	d. c.	L. Cot.	L. Cos.	d.	
8 0	9.1436	89	9.1478	91	10.8522	9.9958	2	0 **82**
10	9.1525	87	9.1569	89	10.8431	9.9956	2	50
20	9.1612	85	9.1658	87	10.8342	9.9954	2	40
30	9.1697	84	9.1745	86	10.8255	9.9952	2	30
40	9.1781	82	9.1831	84	10.8169	9.9950	2	20
50	9.1863	80	9.1915	82	10.8085	9.9948	2	10
9 0	9.1943	79	9.1997	81	10.8003	9.9946	2	0 **81**
10	9.2022	78	9.2078	80	10.7922	9.9944	2	50
20	9.2100	76	9.2158	78	10.7842	9.9942	2	40
30	9.2176	75	9.2236	77	10.7764	9.9940	2	30
40	9.2251	73	9.2313	76	10.7687	9.9938	2	20
50	9.2324	73	9.2389	74	10.7611	9.9936	2	10
10 0	9.2397	71	9.2463	73	10.7537	9.9934	3	0 **80**
10	9.2468	70	9.2536	73	10.7464	9.9931	2	50
20	9.2538	68	9.2609	71	10.7391	9.9929	2	40
30	9.2606	68	9.2680	70	10.7320	9.9927	3	30
40	9.2674	66	9.2750	69	10.7250	9.9924	2	20
50	9.2740	66	9.2819	68	10.7181	9.9922	3	10
11 0	9.2806	64	9.2887	66	10.7113	9.9919	2	0 **79**
10	9.2870	64	9.2953	67	10.7047	9.9917	3	50
20	9.2934	63	9.3020	65	10.6980	9.9914	2	40
30	9.2997	61	9.3085	64	10.6915	9.9912	3	30
40	9.3058	61	9.3149	63	10.6851	9.9909	2	20
50	9.3119	60	9.3212	63	10.6788	9.9907	3	10
12 0	9.3179	59	9.3275	61	10.6725	9.9904	3	0 **78**
10	9.3238	58	9.3336	61	10.6664	9.9901	2	50
20	9.3296	57	9.3397	61	10.6603	9.9899	3	40
30	9.3353	57	9.3458	59	10.6542	9.9896	3	30
40	9.3410	56	9.3517	59	10.6483	9.9893	3	20
50	9.3466	55	9.3576	58	10.6424	9.9890	3	10
13 0	9.3521	54	9.3634	57	10.6366	9.9887	3	0 **77**
10	9.3575	54	9.3691	57	10.6309	9.9884	3	50
20	9.3629	53	9.3748	56	10.6252	9.9881	3	40
30	9.3682	52	9.3804	55	10.6196	9.9878	3	30
40	9.3734	52	9.3859	55	10.6141	9.9875	3	20
50	9.3786	51	9.3914	54	10.6086	9.9872	3	10
14 0	9.3837	50	9.3968	53	10.6032	9.9869	3	0 **76**
10	9.3887	50	9.4021	53	10.5979	9.9866	3	50
20	9.3937	49	9.4074	53	10.5926	9.9863	4	40
30	9.3986	49	9.4127	51	10.5873	9.9859	3	30
40	9.4035	48	9.4178	52	10.5822	9.9856	3	20
50	9.4083	47	9.4230	51	10.5770	9.9853	4	10
15 0	9.4130	47	9.4281	50	10.5719	9.9849	3	0 **75**
10	9.4177	46	9.4331	50	10.5669	9.9846	3	50
20	9.4223	46	9.4381	49	10.5619	9.9843	4	40
30	9.4269	45	9.4430	49	10.5570	9.9839	3	30
40	9.4314	45	9.4479	48	10.5521	9.9836	4	20
50	9.4359	44	9.4527	48	10.5473	9.9832	4	10
16 0	9.4403		9.4575		10.5425	9.9828		0 **74**
	L. Cos.	d.	L. Cot.	d. c.	L. Tan.	L. Sin.	d.	′ °

° ′	L. Sin.	d.	L. Tan.	d. c.	L. Cot.	L. Cos.	d.	
16 0	9.4403	44	9.4575	47	10.5425	9.9828	3	0 **74**
10	9.4447	44	9.4622	47	10.5378	9.9825	4	50
20	9.4491	42	9.4669	47	10.5331	9.9821	4	40
30	9.4533	43	9.4716	46	10.5284	9.9817	3	30
40	9.4576	42	9.4762	46	10.5238	9.9814	4	20
50	9.4618	41	9.4808	45	10.5192	9.9810	4	10
17 0	9.4659	41	9.4853	45	10.5147	9.9806	4	0 **73**
10	9.4700	41	9.4898	45	10.5102	9.9802	4	50
20	9.4741	40	9.4943	44	10.5057	9.9798	4	40
30	9.4781	40	9.4987	44	10.5013	9.9794	4	30
40	9.4821	40	9.5031	44	10.4969	9.9790	4	20
50	9.4861	39	9.5075	43	10.4925	9.9786	4	10
18 0	9.4900	39	9.5118	43	10.4882	9.9782	4	0 **72**
10	9.4939	38	9.5161	42	10.4839	9.9778	4	50
20	9.4977	38	9.5203	42	10.4797	9.9774	4	40
30	9.5015	37	9.5245	42	10.4755	9.9770	5	30
40	9.5052	38	9.5287	42	10.4713	9.9765	4	20
50	9.5090	36	9.5329	41	10.4671	9.9761	4	10
19 0	9.5120	37	9.5370	41	10.4630	9.9757	5	0 **71**
10	9.5163	36	9.5411	40	10.4589	9.9752	4	50
20	9.5199	36	9.5451	40	10.4549	9.9748	5	40
30	9.5235	35	9.5491	40	10.4509	9.9743	4	30
40	9.5270	36	9.5531	40	10.4469	9.9739	5	20
50	9.5306	35	9.5571	40	10.4429	9.9734	4	10
20 0	9.5341	34	9.5611	39	10.4389	9.9730	5	0 **70**
10	9.5375	34	9.5650	39	10.4350	9.9725	4	50
20	9.5409	34	9.5689	38	10.4311	9.9721	5	40
30	9.5443	34	9.5727	39	10.4273	9.9716	5	30
40	9.5477	33	9.5766	38	10.4234	9.9711	5	20
50	9.5510	33	9.5804	38	10.4196	9.9706	4	10
21 0	9.5543	33	9.5842	37	10.4158	9.9702	5	0 **69**
10	9.5576	33	9.5879	38	10.4121	9.9697	5	50
20	9.5609	32	9.5917	37	10.4083	9.9692	5	40
30	9.5641	32	9.5954	37	10.4046	9.9687	5	30
40	9.5673	31	9.5991	37	10.4009	9.9682	5	20
50	9.5704	32	9.6028	36	10.3972	9.9677	5	10
22 0	9.5736	31	9.6064	36	10.3936	9.9672	5	0 **68**
10	9.5767	31	9.6100	36	10.3900	9.9667	6	50
20	9.5798	30	9.6136	36	10.3864	9.9661	5	40
30	9.5828	31	9.6172	36	10.3828	9.9656	5	30
40	9.5859	30	9.6208	35	10.3792	9.9651	5	20
50	9.5889	30	9.6243	36	10.3757	9.9646	6	10
23 0	9.5919	29	9.6279	35	10.3721	9.9640	5	0 **67**
10	9.5948	30	9.6314	34	10.3686	9.9635	6	50
20	9.5978	29	9.6348	35	10.3652	9.9629	5	40
30	9.6007	29	9.6383	34	10.3617	9.9624	6	30
40	9.6036	29	9.6417	35	10.3583	9.9618	5	20
50	9.6065	28	9.6452	34	10.3548	9.9613	6	10
24 0	9.6093		9.6486		10.3514	9.9607		0 **66**
	L. Cos.	d.	L. Cot.	d. c.	L. Tan.	L. Sin.	d.	′ °

° ′	L. Sin.	d.	L. Tan.	d. c.	L. Cot.	L. Cos.	d.	
24 0	9.6093	28	9.6486	34	10.3514	9.9607	5	0 **66**
10	9.6121	28	9.6520	33	10.3480	9.9602	6	50
20	9.6149	28	9.6553	34	10.3447	9.9596	6	40
30	9.6177	28	9.6587	33	10.3413	9.9590	6	30
40	9.6205	27	9.6620	34	10.3380	9.9584	5	20
50	9.6232	27	9.6654	33	10.3346	9.9579	6	10
25 0	9.6259	27	9.6687	33	10.3313	9.9573	6	0 **65**
10	9.6286	27	9.6720	32	10.3280	9.9567	6	50
20	9.6313	27	9.6752	33	10.3248	9.9561	6	40
30	9.6340	26	9.6785	32	10.3215	9.9555	6	30
40	9.6366	26	9.6817	33	10.3183	9.9549	6	20
50	9.6392	26	9.6850	32	10.3150	9.9543	6	10
26 0	9.6418	26	9.6882	32	10.3118	9.9537	7	0 **64**
10	9.6444	26	9.6914	32	10.3086	9.9530	6	50
20	9.6470	25	9.6946	31	10.3054	9.9524	6	40
30	9.6495	26	9.6977	32	10.3023	9.9518	6	30
40	9.6521	25	9.7009	31	10.2991	9.9512	7	20
50	9.6546	24	9.7040	32	10.2960	9.9505	6	10
27 0	9.6570	25	9.7072	31	10 2928	9.9499	7	0 **63**
10	9.6595	25	9.7103	31	10.2897	9.9492	6	50
20	9.6620	24	9.7134	31	10.2866	9.9486	7	40
30	9.6644	24	9.7165	31	10.2835	9.9479	6	30
40	9.6668	24	9.7196	30	10.2804	9.9473	7	20
50	9.6692	24	9.7226	31	10 2774	9.9466	7	10
28 0	9.6716	24	9.7257	30	10.2743	9.9459	6	0 **62**
10	9.6740	23	9.7287	30	10.2713	9.9453	7	50
20	9.6763	24	9.7317	31	10.2683	9.9446	7	40
30	9.6787	23	9.7348	30	10.2652	9.9439	7	30
40	9.6810	23	9.7378	30	10.2622	9.9432	7	20
50	9.6833	23	9.7408	30	10.2592	9 9425	7	10
29 0	9.6856	22	9.7438	29	10.2562	9.9418	7	0 **61**
10	9.6878	23	9.7467	30	10.2533	9.9411	7	50
20	9.6901	22	9.7497	29	10.2503	9.9404	7	40
30	9.6923	23	9.7526	30	10.2474	9.9397	7	30
40	9.6946	22	9.7556	29	10.2444	9.9390	7	20
50	9.6968	22	9.7585	29	10.2415	9.9383	8	10
30 0	9.6990	22	9.7614	30	10.2386	9.9375	7	0 **60**
10	9.7012	21	9.7644	29	10.2356	9.9368	7	50
20	9.7033	22	9.7673	28	10.2327	9.9361	8	40
30	9.7055	21	9.7701	29	10.2299	9.9353	7	30
40	9.7076	21	9.7730	29	10.2270	9.9346	8	20
50	9.7097	21	9.7759	29	10.2241	9.9338	7	10
31 0	9.7118	21	9.7788	28	10.2212	9.9331	8	0 **59**
10	9.7139	21	9.7816	29	10.2184	9.9323	8	50
20	9.7160	21	9.7845	28	10.2155	9.9315	7	40
30	9.7181	20	9.7873	29	10.2127	9.9308	8	30
40	9.7201	21	9.7902	28	10.2098	9.9300	8	20
50	9.7222	20	9.7930	28	10.2070	9.9292	8	10
32 0	9.7242		9.7958		10.2042	9.9284		0 **58**
	L. Cos.	**d.**	**L. Cot.**	**d. c.**	**L. Tan.**	**L. Sin.**	**d.**	′ °

TABLE II.

° ′	L. Sin.	d.	L. Tan.	d. c.	L. Cot.	L. Cos.	d.	
32 0	9.7242	20	9.7958	28	10.2042	9.9284	8	0 **58**
10	9.7262	20	9.7986	28	10.2014	9.9276	8	50
20	9.7282	20	9.8014	28	10.1986	9.9268	8	40
30	9.7302	20	9.8042	28	10.1958	9.9260	8	30
40	9.7322	20	9.8070	27	10.1930	9.9252	8	20
50	9.7342	19	9.8097	28	10.1903	9.9244	8	10
33 0	9.7361	19	9.8125	28	10.1875	9.9236	8	0 **57**
10	9.7380	20	9.8153	27	10.1847	9.9228	9	50
20	9.7400	19	9.8180	28	10.1820	9.9219	8	40
30	9.7419	19	9.8208	27	10.1792	9.9211	8	30
40	9.7438	19	9.8235	28	10.1765	9.9203	9	20
50	9.7457	19	9.8263	27	10.1737	9.9194	8	10
34 0	9.7476	18	9.8290	27	10.1710	9.9186	9	0 **56**
10	9.7494	19	9.8317	27	10.1683	9.9177	8	50
20	9.7513	18	9.8344	27	10.1656	9.9169	9	40
30	9.7531	19	9.8371	27	10.1629	9.9160	9	30
40	9.7550	18	9.8398	27	10.1602	9.9151	9	20
50	9.7568	18	9.8425	27	10.1575	9.9142	8	10
35 0	9.7586	18	9.8452	27	10.1548	9.9134	9	0 **55**
10	9.7604	18	9.8479	27	10.1521	9.9125	9	50
20	9.7622	18	9.8506	27	10.1494	9.9116	9	40
30	9.7640	17	9.8533	26	10.1467	9.9107	9	30
40	9.7657	18	9.8559	27	10.1441	9.9098	9	20
50	9.7675	17	9.8586	27	10.1414	9.9089	9	10
36 0	9.7692	18	9.8613	26	10.1387	9.9080	10	0 **54**
10	9.7710	17	9.8639	27	10.1361	9.9070	9	50
20	9.7727	17	9.8666	26	10.1334	9.9061	9	40
30	9.7744	17	9.8692	26	10.1308	9.9052	10	30
40	9.7761	17	9.8718	27	10.1282	9.9042	9	20
50	9.7778	17	9.8745	26	10.1255	9.9033	10	10
37 0	9.7795	16	9.8771	26	10.1229	9.9023	9	0 **53**
10	9.7811	17	9.8797	27	10.1203	9.9014	10	50
20	9.7828	16	9.8824	26	10.1176	9.9004	9	40
30	9.7844	17	9.8850	26	10.1150	9.8995	10	30
40	9.7861	16	9.8876	26	10.1124	9.8985	10	20
50	9.7877	16	9.8902	26	10.1098	9.8975	10	10
38 0	9.7893	17	9.8928	26	10.1072	9.8965	10	0 **52**
10	9.7910	16	9.8954	26	10.1046	9.8955	10	50
20	9.7926	15	9.8980	26	10.1020	9.8945	10	40
30	9.7941	16	9.9006	26	10.0994	9.8935	10	30
40	9.7957	16	9.9032	26	10.0968	9.8925	10	20
50	9.7973	16	9.9058	26	10.0942	9.8915	10	10
39 0	9.7989	15	9.9084	26	10.0916	9.8905	10	0 **51**
10	9.8004	16	9.9110	25	10.0890	9.8895	11	50
20	9.8020	15	9.9135	26	10.0865	9.8884	10	40
30	9.8035	15	9.9161	26	10.0839	9.8874	10	30
40	9.8050	16	9.9187	25	10.0813	9.8864	11	20
50	9.8066	15	9.9212	26	10.0788	9.8853	10	10
40 0	9.8081		9.9238		10.0762	9.8843		0 **50**
	L. Cos.	d.	L. Cot.	d. c.	L. Tan.	L. Sin.	d.	′ °

° ′	L. Sin.	d.	L. Tan.	d. c.	L. Cot.	L. Cos.	d.	
40 0	9.8081	15	9.9238	26	10.0762	9.8843	11	0 **50**
10	9.8096	15	9.9264	25	10.0736	9.8832	11	50
20	9.8111	14	9.9289	26	10.0711	9.8821	11	40
30	9.8125	15	9.9315	26	10.0685	9.8810	10	30
40	9.8140	15	9.9341	25	10.0659	9.8800	11	20
50	9.8155	14	9.9366	26	10.0634	9.8789	11	10
41 0	9.8169	15	9.9392	25	10.0608	9.8778	11	0 **49**
10	9.8184	14	9.9417	26	10.0583	9.8767	11	50
20	9.8198	15	9.9443	25	10.0557	9.8756	11	40
30	9.8213	14	9.9468	26	10.0532	9.8745	12	30
40	9.8227	14	9.9494	25	10.0506	9.8733	11	20
50	9.8241	14	9.9519	25	10.0481	9.8722	11	10
42 0	9.8255	14	9.9544	26	10.0456	9.8711	12	0 **48**
10	9.8269	14	9.9570	25	10.0430	9.8699	11	50
20	9.8283	14	9.9595	26	10.0405	9.8688	12	40
30	9.8297	14	9.9621	25	10.0379	9.8676	11	30
40	9.8311	13	9.9646	25	10.0354	9.8665	12	20
50	9.8324	14	9.9671	26	10.0329	9.8653	12	10
43 0	9.8338	13	9.9697	25	10.0303	9.8641	12	0 **47**
10	9.8351	14	9.9722	25	10.0278	9.8629	11	50
20	9.8365	13	9.9747	25	10.0253	9.8618	12	40
30	9.8378	13	9.9772	26	10.0228	9.8606	12	30
40	9.8391	14	9.9798	25	10.0202	9.8594	12	20
50	9.8405	13	9.9823	25	10.0177	9.8582	13	10
44 0	9.8418	13	9.9848	26	10.0152	9.8569	12	0 **46**
10	9.8431	13	9.9874	25	10.0126	9.8557	12	50
20	9.8444	13	9.9899	25	10.0101	9.8545	13	40
30	9.8457	12	9.9924	25	10.0076	9.8532	12	30
40	9.8469	13	9.9949	26	10.0051	9.8520	13	20
50	9.8482	13	9.9975	25	10.0025	9.8507	12	10
45 0	9.8495		10.0000		10.0000	9.8495		0 **45**
	L. Cos.	d.	L. Cot.	d. c.	L. Tan.	L. Sin.	d.	′ °

TABLE III.

NATURAL TRIGONOMETRIC FUNCTIONS.

$R = 1.$

0°.

° ′	sin	tan	cot	cos	
0 0	0.00000	0.00000	infinite	1.00000	60
1	0.00029	0.00029	3437.75	1.00000	59
2	0.00058	0.00058	1718.87	1.00000	58
3	0.00087	0.00087	1145.92	1.00000	57
4	0.00116	0.00116	859.436	1.00000	56
5	0.00145	0.00145	687.549	1.00000	55
6	0.00175	0.00175	572.957	1.00000	54
7	0.00204	0.00204	491.106	1.00000	53
8	0.00233	0.00233	429.718	1.00000	52
9	0.00262	0.00262	381.971	1.00000	51
10	0.00291	0.00291	343.774	1.00000	50
11	0.00320	0.00320	312.521	0.99999	49
12	0.00349	0.00349	286.478	0.99999	48
13	0.00378	0.00378	264.441	0.99999	47
14	0.00407	0.00407	245.552	0.99999	46
15	0.00436	0.00436	229.182	0.99999	45
16	0.00465	0.00465	214.858	0.99999	44
17	0.00495	0.00495	202.219	0.99999	43
18	0.00524	0.00524	190.984	0.99999	42
19	0.00553	0.00553	180.932	0.99998	41
20	0.00582	0.00582	171.885	0.99998	40
21	0.00611	0.00611	163.700	0.99998	39
22	0.00640	0.00640	156.259	0.99998	38
23	0.00669	0.00669	149.465	0.99998	37
24	0.00698	0.00698	143.237	0.99998	36
25	0.00727	0.00727	137.507	0.99997	35
26	0.00756	0.00756	132.219	0.99997	34
27	0.00785	0.00785	127.321	0.99997	33
28	0.00814	0.00814	122.774	0.99997	32
29	0.00844	0.00844	118.540	0.99996	31
30	0.00873	0.00873	114.589	0.99996	30
31	0.00902	0.00902	110.892	0.99996	29
32	0.00931	0.00931	107.426	0.99996	28
33	0.00960	0.00960	104.171	0.99995	27
34	0.00989	0.00989	101.107	0.99995	26
35	0.01018	0.01018	98.2179	0.99995	25
36	0.01047	0.01047	95.4895	0.99995	24
37	0.01076	0.01076	92.9085	0.99994	23
38	0.01105	0.01105	90.4633	0.99994	22
39	0.01134	0.01134	88.1436	0.99994	21
40	0.01164	0.01164	85.9398	0.99993	20
41	0.01193	0.01193	83.8435	0.99993	19
42	0.01222	0.01222	81.8470	0.99993	18
43	0.01251	0.01251	79.9434	0.99992	17
44	0.01280	0.01280	78.1263	0.99992	16
45	0.01309	0.01309	76.3900	0.99991	15
46	0.01338	0.01338	74.7292	0.99991	14
47	0.01367	0.01367	73.1390	0.99991	13
48	0.01396	0.01396	71.6151	0.99990	12
49	0.01425	0.01425	70.1533	0.99990	11
50	0.01454	0.01454	68.7501	0.99989	10
51	0.01483	0.01483	67.4019	0.99989	9
52	0.01513	0.01513	66.1055	0.99989	8
53	0.01542	0.01542	64.8580	0.99988	7
54	0.01571	0.01571	63.6567	0.99988	6
55	0.01600	0.01600	62.4992	0.99987	5
56	0.01629	0.01629	61.3829	0.99987	4
57	0.01658	0.01658	60.3058	0.99986	3
58	0.01687	0.01687	59.2659	0.99986	2
59	0.01716	0.01716	58.2612	0.99985	1 89
	cos	cot	tan	sin	′ °

89°.

1° — 8°.

° ′	sin	tan	cot	cos	′ °
1 0	0.01745	0.01746	57.2900	0.99985	0 **89**
10	0.02036	0.02036	49.1039	0.99979	50
20	0.02327	0.02328	42.9641	0.99973	40
30	0.02618	0.02619	38.1885	0.99966	30
40	0.02908	0.02910	34.3678	0.99958	20
50	0.03199	0.03201	31.2416	0.99949	10
2 0	0.03490	0.03492	28.6363	0.99939	0 **88**
10	0.03781	0.03783	26.4316	0.99929	50
20	0.04071	0.04075	24.5418	0.99917	40
30	0.04362	0.04366	22.9038	0.99905	30
40	0.04653	0.04658	21.4704	0.99892	20
50	0.04943	0.04949	20.2056	0.99878	10
3 0	0.05234	0.05241	19.0811	0.99863	0 **87**
10	0.05524	0.05533	18.0750	0.99847	50
20	0.05814	0.05824	17.1693	0.99831	40
30	0.06105	0.06116	16.3499	0.99813	30
40	0.06395	0.06408	15.6048	0.99795	20
50	0.06685	0.06700	14.9244	0.99776	10
4 0	0.06976	0.06993	14.3007	0.99756	0 **86**
10	0.07266	0.07285	13.7267	0.99736	50
20	0.07556	0.07578	13.1969	0.99714	40
30	0.07846	0.07870	12.7062	0.99692	30
40	0.08136	0.08163	12.2505	0.99668	20
50	0.08426	0.08456	11.8262	0.99644	10
5 0	0.08716	0.08749	11.4301	0.99619	0 **85**
10	0.09005	0.09042	11.0594	0.99594	50
20	0.09295	0.09335	10.7119	0.99567	40
30	0.09585	0.09629	10.3854	0.99540	30
40	0.09874	0.09923	10.0780	0.99511	20
50	0.10164	0.10216	9.78817	0.99482	10
6 0	0.10453	0.10510	9.51436	0.99452	0 **84**
10	0.10742	0.10805	9.25530	0.99421	50
20	0.11031	0.11099	9.00983	0.99390	40
30	0.11320	0.11394	8.77689	0.99357	30
40	0.11609	0.11688	8.55555	0.99324	20
50	0.11898	0.11983	8.34496	0.99290	10
7 0	0.12187	0.12278	8.14435	0.99255	0 **83**
10	0.12476	0.12574	7.95302	0.99219	50
20	0.12764	0.12869	7.77035	0.99182	40
30	0.13053	0.13165	7.59575	0.99144	30
40	0.13341	0.13461	7.42871	0.99106	20
50	0.13629	0.13758	7.26873	0.99067	10
8 0	0.13917	0.14054	7.11537	0.99027	0 **82**
10	0.14205	0.14351	6.96823	0.98986	50
20	0.14493	0.14648	6.82694	0.98944	40
30	0.14781	0.14945	6.69116	0.98902	30
40	0.15069	0.15243	6.56055	0.98858	20
50	0.15356	0.15540	6.43484	0.98814	10
9 0	0.15643	0.15838	6.31375	0.98769	0 **81**
° ′	cos	cot	tan	sin	′ °

81° — 88°.

TABLE III.

9° — 16°.

° ′	sin	tan	cot	cos	′ °
9 0	0.15643	0.15838	6.31375	0.98769	0 **81**
10	0.15931	0.16137	6.19703	0.98723	50
20	0.16218	0.16435	6.08444	0.98676	40
30	0.16505	0.16734	5.97576	0.98629	30
40	0.16792	0.17033	5.87080	0.98580	20
50	0.17078	0.17333	5.76937	0.98531	10
10 0	0.17365	0.17633	5.67128	0.98481	0 **80**
10	0.17651	0.17933	5.57638	0.98430	50
20	0.17937	0.18233	5.48451	0.98378	40
30	0.18224	0.18534	5.39552	0.98325	30
40	0.18509	0.18835	5.30928	0.98272	20
50	0.18795	0.19136	5.22566	0.98218	10
11 0	0.19081	0.19438	5.14455	0.98163	0 **79**
10	0.19366	0.19740	5.06584	0.98107	50
20	0.19652	0.20042	4.98940	0.98050	40
30	0.19937	0.20345	4.91516	0.97992	30
40	0.20222	0.20648	4.84300	0.97934	20
50	0.20507	0.20952	4.77286	0.97875	10
12 0	0.20791	0.21256	4.70463	0.97815	0 **78**
10	0.21076	0.21560	4.63825	0.97754	50
20	0.21360	0.21864	4.57363	0.97692	40
30	0.21644	0.22169	4.51071	0.97630	30
40	0.21928	0.22475	4.44942	0.97566	20
50	0.22212	0.22781	4.38969	0.97502	10
13 0	0.22495	0.23087	4.33148	0.97437	0 **77**
10	0.22778	0.23393	4.27471	0.97371	50
20	0.23062	0.23700	4.21933	0.97304	40
30	0.23345	0.24008	4.16530	0.97237	30
40	0.23627	0.24316	4.11256	0.97169	20
50	0.23910	0.24624	4.06107	0.97100	10
14 0	0.24192	0.24933	4.01078	0.97030	0 **76**
10	0.24474	0.25242	3.96165	0.96959	50
20	0.24756	0.25552	3.91364	0.96887	40
30	0.25038	0.25862	3.86671	0.96815	30
40	0.25320	0.26172	3.82083	0.96742	20
50	0.25601	0.26483	3.77595	0.96667	10
15 0	0.25882	0.26795	3.73205	0.96593	0 **75**
10	0.26163	0.27107	3.68909	0.96517	50
20	0.26443	0.27419	3.64705	0.96440	40
30	0.26724	0.27732	3.60588	0.96363	30
40	0.27004	0.28046	3.56557	0.96285	20
50	0.27284	0.28360	3.52609	0.96206	10
16 0	0.27564	0.28675	3.48741	0.96126	0 **74**
10	0.27843	0.28990	3.44951	0.96046	50
20	0.28123	0.29305	3.41236	0.95964	40
30	0.28402	0.29621	3.37594	0.95882	30
40	0.28680	0.29938	3.34023	0.95799	20
50	0.28959	0.30255	3.30521	0.95715	10
17 0	0.29237	0.30573	3.27085	0.95630	0 **73**
° ′	cos	cot	tan	sin	′ °

73° — 80°.

17° — 23°.

° ′	**sin**	**tan**	**cot**	**cos**	′ °
17 0	0.29237	0.30573	3.27085	0.95630	0 **73**
10	0.29515	0.30891	3.23714	0.95545	50
20	0.29793	0.31210	3.20406	0.95459	40
30	0.30071	0.31530	3.17159	0.95372	30
40	0.30348	0.31850	3.13972	0.95284	20
50	0.30625	0.32171	3.10842	0.95195	10
18 0	0.30902	0.32492	3.07768	0.95106	0 **72**
10	0.31178	0.32814	3.04749	0.95065	50
20	0.31454	0.33136	3.01783	0.94924	40
30	0.31730	0.33460	2.98868	0.94832	30
40	0.32006	0.33783	2.96004	0.94740	20
50	0.32282	0.34108	2.93189	0.94646	10
19 0	0.32557	0.34433	2.90421	0.94552	0 **71**
10	0.32832	0.34758	2.87700	0.94457	50
20	0.33106	0.35085	2.85023	0.94361	40
30	0.33381	0.35412	2.82391	0.94264	30
40	0.33655	0.35740	2.79802	0.94167	20
50	0.33929	0.36068	2.77254	0.94068	10
20 0	0.34202	0.36397	2.74748	0.93969	0 **70**
10	0.34475	0.36727	2.72281	0.93869	50
20	0.34748	0.37057	2.69853	0.93769	40
30	0.35021	0.37388	2.67462	0.93667	30
40	0.35293	0.37720	2.65109	0.93565	20
50	0.35565	0.38053	2.62791	0.93462	10
21 0	0.35837	0.38386	2.60509	0.93358	0 **69**
10	0.36108	0.38721	2.58261	0.93253	50
20	0.36379	0.39055	2.56046	0.93148	40
30	0.36650	0.39391	2.53865	0.93042	30
40	0.36921	0.39727	2.51715	0.92935	20
50	0.37191	0.40065	2.49597	0.92827	10
22 0	0.37461	0.40403	2.47509	0.92718	0 **68**
10	0.37730	0.40741	2.45451	0.92609	50
20	0.37999	0.41081	2.43422	0.92499	40
30	0.38268	0.41421	2.41421	0.92388	30
40	0.38537	0.41763	2.39449	0.92276	20
50	0.38805	0.42105	2.37504	0.92164	10
23 0	0.39073	0.42447	2.35585	0.92050	0 **67**
10	0.39341	0.42791	2.33693	0.91936	50
20	0.39608	0.43136	2.31826	0.91822	40
30	0.39875	0.43481	2.29984	0.91706	30
40	0.40141	0.43828	2.28167	0.91590	20
50	0.40408	0.44175	2.26374	0.91472	10
24 0	0.40674	0.44523	2.24604	0.91355	0 **66**
° ′	**cos**	**cot**	**tan**	**sin**	′ °

66° — 73°.

TABLE III.

24° — 30°.

° ′	sin	tan	cot	cos	′ °
24 0	0.40674	0.44523	2.24604	0.91355	0 **66**
10	0.40939	0.44872	2.22857	0.91236	50
20	0.41204	0.45222	2.21132	0.91116	40
30	0.41469	0.45573	2.19430	0.90996	30
40	0.41734	0.45924	2.17749	0.90875	20
50	0.41998	0.46277	2.16090	0.90753	10
25 0	0.42262	0.46631	2.14451	0.90631	0 **65**
10	0.42525	0.46985	2.12832	0.90507	50
20	0.42788	0.47341	2.11233	0.90383	40
30	0.43051	0.47698	2.09654	0.90259	30
40	0.43313	0.48055	2.08094	0.90133	20
50	0.43575	0.48414	2.06553	0.90007	10
26 0	0.43837	0.48773	2.05030	0.89879	0 **64**
10	0.44098	0.49134	2.03526	0.89752	50
20	0.44359	0.49495	2.02039	0.89623	40
30	0.44620	0.49858	2.00569	0.89493	30
40	0.44880	0.50222	1.99116	0.89363	20
50	0.45140	0.50587	1.97680	0.89232	10
27 0	0.45399	0.50953	1.96261	0.89101	0 **63**
10	0.45658	0.51319	1.94858	0.88968	50
20	0.45917	0.51688	1.93470	0.88835	40
30	0.46175	0.52057	1.92098	0.88701	30
40	0.46433	0.52427	1.90741	0.88566	20
50	0.46690	0.52798	1.89400	0.88431	10
28 0	0.46947	0.53171	1.88073	0.88295	0 **62**
10	0.47204	0.53545	1.86760	0.88158	50
20	0.47460	0.53920	1.85462	0.88020	40
30	0.47716	0.54296	1.84177	0.87882	30
40	0.47971	0.54673	1.82906	0.87743	20
50	0.48226	0.55051	1.81649	0.87603	10
29 0	0.48481	0.55431	1.80405	0.87462	0 **61**
10	0.48735	0.55812	1.79174	0.87321	50
20	0.48989	0.56194	1.77955	0.87178	40
30	0.49242	0.56577	1.76749	0.87036	30
40	0.49495	0.56962	1.75556	0.86892	20
50	0.49748	0.57348	1.74375	0.86748	10
30 0	0.50000	0.57735	1.73205	0.86603	0 **60**
10	0.50252	0.58124	1.72047	0.86457	50
20	0.50503	0.58513	1.70901	0.86310	40
30	0.50754	0.58904	1.69766	0.86163	30
40	0.51004	0.59297	1.68643	0.86015	20
50	0.51254	0.59691	1.67530	0.85866	10
31 0	0.51504	0.60086	1.66428	0.85717	0 **59**
° ′	cos	cot	tan	sin	′ °

59° — 65°.

31° — 37°.

° ′	sin	tan	cot	cos	′ °
31 0	0.51504	0.60086	1.66428	0.85717	0 **59**
10	0.51753	0.60483	1.65337	0.85567	50
20	0 52002	0.60881	1.64256	0.85416	40
30	0.52250.	0.61280	1.63185	0.85264	30
40	0.52498	0.61681	1.62125	0.85112	20
50	0.52745	0.62083	1.61074	0.84959	10
32 0	0.52992	0 62487	1.60033	0.84805	0 **58**
10	0.53238	0.62892	1.59002	0.84650	50
20	0.53484	0.63299	1.57981	0.84495	40
30	0.53730	0.63707	1.56969	0.84339	30
40	0.53975	0.64117	1.55966	0.84182	20
50	0.54220	0.64528	1.54972	0.84025	10
33 0	0.54464	0.64941	1.53986	0.83867	0 **57**
10	0.54708	0.65355	1.53010	0.83708	50
20	0.54951	0.65771	1.52043	0.83549	40
30	0.55194	0.66189	1.51084	0.83389	30
40	0.55436	0.66608	1.50133	0.83228	20
50	0.55678	0.67028	1.49190	0.83066	10
34 0	0.55919	0.67451	1.48256	0.82904	0 **56**
10	0.56160	0.67875	1.47330	0.82741	50
20	0.56401	0.68301	1.46411	0.82577	40
30	0.56641	0.68728	1.45501	0.82413	30
40	0.56880	0.69157	1.44598	0.82248	20
50	0.57119	0.69588	1.43703	0.82082	10
35 0	0.57358	0.70021	1.42815	0.81915	0 **55**
10	0.57596	0.70455	1.41934	0.81748	50
20	0.57833	0.70891	1.41061	0.81580	40
30	0.58070	0.71329	1.40195	0.81412	30
40	0.58307	0.71769	1.39336	0.81242	20
50	0.58543	0.72211	1.38484	0.81072	10
36 0	0.58779	0.72654	1.37638	0.80902	0 **54**
10	0.59014	0.73100	1.36800	0.80730	50
20	0.59248	0.73547	1.35968	0.80558	40
30	0.59482	0.73996	1.35142	0.80386	30
40	0.59716	0.74447	1.34323	0.80212	20
50	0.59949	0.74900	1.33511	0.80038	10
37 0	0.60182	0.75355	1.32704	0.79864	0 **53**
10	0.60414	0.75812	1.31904	0.79688	50
20	0.60645	0.76272	1.31110	0.79512	40
30	0.60876	0.76733	1.30323	0.79335	30
40	0.61107	0.77196	1.29541	0.79158	20
50	0.61337	0.77661	1.28764	0.78980	10
38 0	0.61566	0.78129	1.27994	0.78801	0 **52**
° ′	cos	cot	tan	sin	′ °

52° — 58°.

38° — 44°.

° ′	**sin**	**tan**	**cot**	**cos**	′ °
38 0	0.61566	0.78129	1.27994	0.78801	0 **52**
10	0.61795	0.78598	1.27230	0.78622	50
20	0.62024	0.79070	1.26471	0.78442	40
30	0.62251	0.79544	1.25717	0.78261	30
40	0.62479	0.80020	1.24969	0.78079	20
50	0.62706	0.80498	1.24227	0.77897	10
39 0	0.62932	0.80978	1.23490	0.77715	0 **51**
10	0.63158	0.81461	1.22758	0.77531	50
20	0.63383	0.81946	1.22031	0.77347	40
30	0.63608	0.82434	1.21310	0.77162	30
40	0.63832	0.82923	1.20593	0.76977	20
50	0.64056	0.83415	1.19882	0.76791	10
40 0	0.64279	0 83910	1.19175	0.76604	0 **50**
10	0.64501	0.84407	1.18474	0.76417	50
20	0.64723	0.84906	1.17777	0.76229	40
30	0.64945	0.85408	1.17085	0.76041	30
40	0.65166	0.85912	1.16398	0.75851	20
50	0.65386	0.86419	1.15715	0.75661	10
41 0	0.65606	0.86929	1.15037	0.75471	0 **49**
10	0.65825	0.87441	1.14363	0.75280	50
20	0.66044	0.87955	1.13694	0.75088	40
30	0.66262	0.88473	1.13029	0.74896	30
40	0.66480	0.88992	1.12369	0.74703	20
50	0.66697	0.89515	1.11713	0.74509	10
42 0	0.66913	0.90040	1.11061	0.74314	0 **48**
10	0.67129	0.90569	1.10414	c.74120	50
20	0.67344	0.91099	1.09770	0.73924	40
30	0.67559	0.91633	1.09131	0.73728	30
40	0.67773	0.92170	1.08496	0.73531	20
50	0.67987	0.92709	1.07864	0.73333	10
43 0	0.68200	0.93252	1.07237	0.73135	0 **47**
10	0.68412	0.93797	1.06613	0.72937	50
20	0.68624	0.94345	1.05994	0.72737	40
30	0.68835	0.94896	1.05378	0.72537	30
40	0.69046	0.95451	1.04766	0.72337	20
50	0.69256	0.96008	1.04158	0.72136	10
44 0	0.69466	0.96569	1.03553	0.71934	0 **46**
10	c.69675	0.97133	1.02952	0.71732	50
20	0.69883	0.97700	1.02355	0.71529	40
30	0 70091	0.98270	1.01761	0.71325	30
40	0.70298	0.98843	1.01170	0.71121	20
50	0.70505	0.99420	1.00583	0.70916	10
45 0	0.70711	1.00000	1.00000	0.70711	0 **45**
° ′	**cos**	**cot**	**tan**	**sin**	′ °

45° — 51°.

TABLE IV.

USEFUL NUMBERS AND THEIR LOGARITHMS.

	Numbers.	**Logarithms.**
Circumference of circle to diameter 1 .	3.14159265359	0.4971499
Area of circle to radius 1	3.14159265359	0.4971499
Surface of sphere to diameter 1 . . .	3.14159265359	0.4971499
Circumference of circle to radius 1 . .	6.28318530718	0.7981799
Solid contents of sphere to diameter 1 .	0.52359877560	$\bar{1}.7189986$
Solid contents of sphere to radius 1 . .	4.18879020479	0.6220886
$\pi^2 = (3.14159265359)^2$	9.86960440109	0.9942997
$\sqrt{\pi} = \sqrt{(3.14159265359)}$	1.77245385091	0.2485750
$\frac{\pi}{360} = \frac{1}{360}(3.14159265359)$	0.00872664626	$\bar{3}.9408474$
$\frac{\pi}{24} = \frac{1}{24}(3.14159265359)$	0 13089969390	$\bar{1}.1169387$
Area of circle to diameter 1	0.78539816340	$\bar{1}.8950899$
360 degrees expressed in seconds . . .	1296000	6.1126050
Arc, equal to radius, in degrees . . .	57.29578	1.7581226
" " " minutes . . .	3437.74677	3.5362739
" " " seconds . . .	206264.8	5.3144251
Length of arc of 1″ = sine of 1″ to R = 1	0.000004848	$\bar{6}.6855749$
" " 2″ = " 2″ "	0.000009696	$\bar{6}.9866049$
" " 3″ = " 3″ "	0.000014544	$\bar{5}.1626961$
" " 1′ = " 1′ "	0.000290888	$\bar{4}.4637261$
Arc of 1° to radius = 1	0.0174532930	$\bar{2}.2418774$
Sine of 1° to radius = 1	0.0174524064	$\bar{2}.2418553$
Equatorial diameter of the earth in miles	7924	3.8989445
Circumference of the equator in miles .	24893.98	4.3960944
Sun's horizontal parallax	8″.848	0.9468451
Logarithmic sine = *log* tan of 8″.848 .		5.6324200

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